A Closer Look at Infinity

Summer@Brown 2020

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Contents

Chapter 1

Introduction

Suppose there are 100 rooms and each room has the exact same infinite sequence of boxes with each box containing a real valued number. There are 100 people and each person is assigned to one of the 100 rooms. After everyone has entered their room, each person must guess what number is in at least one of the boxes. Each person may open as many boxes as they want, even infinitely many but must leave at least one box unopened. Before everyone enters their room, the group of people may decide on a strategy. The challenge is to find a strategy such that 99 people correctly guess what number is in at least one box.

The goal of these notes is to explain how to solve the *100 Rooms Problem.* There are only three concepts from higher level mathematics that are needed to solve this problem, namely equivalence classes, convergence of sequences, and the axiom of choice. The crux of the solution is that the group of people must use the axiom of choice to guide their guess. This axiom has upset mathematicians for decades due to some strange results that have been proven. One such example is the Banach-Tarski Paradox which states that a sphere can be cut into 5 pieces that can be reassembled in a way that creates two spheres. However, even stranger results can be proven if we choose to reject this axiom. The primary focus is to explain how to solve the 100 Rooms Problem, but these notes also include some historical context and the story of controversy behind the infamous axiom of choice.

Although there are only a few concepts that are necessary to solve the 100 Rooms Problem, these notes include several related topics such as mathematical induction, sizes of infinity, the Hilbert Hotel Paradox, and metric spaces. The underlying goal is to introduce students to higher level mathematics in an engaging and interesting manner. The intended audience is high school students, so these notes have been written to be entirely self contained and each topic is kept at an elementary level. There are problems at the end of each chapter which may be assigned as homework or group work to reinforce each concept. Some of these problems are directly used to solve the 100 Rooms Problem and are indicated by the symbol $(\star)$. There are additional logic problems in Chapter 4 with some being quite difficult, but they are intended to be assigned as group work. By the end of the course, students will have solved several parts of the solution to the 100 Rooms Problem and should be able to understand the entire solution. This is a challenging problem but working through each step of the solution helps students develop abstract problem solving skills and encourages creative thinking. I hope these notes makes students more interested in math and gives them a fresh perspective on the kinds of problems they can solve with it. Moreover, that mathematics is a very active field and there are many open problems that these students could solve someday.

Chapter 2

Set Theory

2.1 Basic Concepts and Notation

One of the most fundamental and useful objects in mathematics is a set. The everyday use of the word *set* could refer to something like a dinnerware set which contains plates, cups, and silverware or a makeup set containing lipstick, eyeshadow, and mascara. Similarly, a mathematical set is a collection of objects which are referred to as *elements* or *members*. This term has a precise mathematical definition, but it's simple and aligns with our intuition of this word outside of mathematics.

Definition 1. *A set is an unordered collection of unique objects.*

When we say that a set is *unordered*, this means that the elements can be written in any particular order without changing the set. For example, suppose that $X = \{a, b, c\}$ is a set, then we can also write this set as $X = \{b, c, a\}$ or $X = \{c, a, b\}$. The second characteristic of a set is that each element is *unique*, meaning that an element can only appear in a set once. When an object is contained in a set, the notation for saying this is $a \in X$ and $a \notin X$ means that an object is not contained in a set.

Definition 2. *A set V is a subset of another set U if for every $x \in V$, then $x \in U$.*

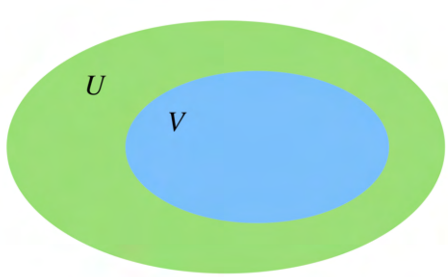


Figure 2.1: Illustration of a subset

The notation for saying that "V is a subset of U" is $V \subset U$. Sometimes it is useful to define a subset in the following manner: $V = \{x \in U : x \text{ satisfies some condition}\}$, which is read as "$V$ is the set of all x contained in U such that x satisfies some condition".

Example 1. *Let $U = \{1, 2, 3, 4, \ldots\}$ be a set and define $V = \{x \in U : x \text{ is an even number}\}$, then the subset V can also be written as $V = \{2, 4, 6, \ldots\}$.*

A set can contain any kind of objects, regardless of whether the objects are concrete or abstract. For example, we can define U to be the set of our favorite numbers or books contained in the SciLi or fish swimming in the Narragansett Bay.

It is also useful to work with *abstract* sets, which means that the elements in the set are not specified. In Definition 1, the phrase "$x \in U$" means that x is a variable which could represent anything. In some some instances, it is useful to define a set as $X = \{x_1, x_2, x_3\}$ where the subscript is used to label each element. The purpose of working with abstract sets is that any result proven for an abstract set can be applied to any particular set. We will use abstract sets from time to time, but there a number of common sets that will be used frequently throughout these notes.

Definition 3. *Common Sets:*

1. ***Empty Set:*** $\emptyset = \{\}$
2. ***Natural Numbers:*** $\mathbb{N} = \{1, 2, 3, \ldots\}$
3. ***Integers:*** $\mathbb{Z} = \{0, 1, 2, 3, \ldots\} \cup \{-1, -2, -3, \ldots\}$
4. ***Rationals:*** $\mathbb{Q} = \{p/q \in \mathbb{R} : p, q \in \mathbb{Z}\}$
5. ***Irrationals:*** $\mathbb{I} = \{x \in \mathbb{R} : x \notin \mathbb{Q}\}$
6. ***Reals:*** $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$
7. ***Power Set:*** *$\mathcal{P}(X)$ is the set of all subsets of some set X.*

Example 2. *Let $X = \{a, b, c\}$, then the power set is*

$$\mathcal{P}(X) = \big\{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\big\}$$

which is referred to as a collection of sets.

Next, let's define some useful operations that can be used to create new sets from old sets. Set operations are analogous to the four operations of arithmetic; namely addition, subtraction, multiplication, and division. Similarly, there are several useful operations that can be used on sets.

Definition 4. *Let U and V be two sets and for the sake of illustration let $A = \{x_1, x_2, x_3\}$ and $B = \{x_2, x_3, x_4\}$, now define the following set operations.*

1. ***Union:*** *Let $U \cup V$ denote the union of two sets such that the union is the set of all elements contained in U and V.*

   Example: $A \cup B = \{x_1, x_2, x_3, x_4\}$

2. ***Intersection*** *Let $U \cap V$ denote the intersection between two sets such that the intersection is the set of elements contained in both U and V.*

   Example: $A \cap B = \{x_2, x_3\}$

3. ***Set Difference*** *Let $U \backslash V$ denote set difference with the difference being the set of elements contained in U but not in V.*

   Example: $A \backslash B = \{x_1\}$

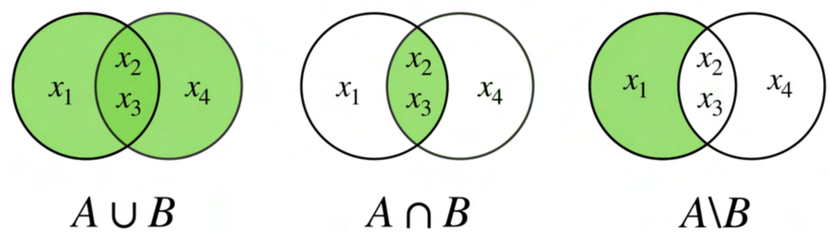


Figure 2.2: Set operations

We can add elements to a set by using the union operation and remove elements with the set difference operation. The last operation is the intersection of two sets which is the set of elements contained in both sets. Another type of operation that we will use is the *cartesian product* between two sets. The main idea of this operation is that it generates all possible pairs of elements from two different sets.

Definition 5. *Let U and V be two sets and define the cartesian product to be $U \times V = \{(u, v) : u \in U, v \in V\}$.*

As an elementary school student, you probably only worked with numbers on the real line $\mathbb{R}$. But in about middle school, you may have learned about the xy-plane where every point can be written as (x, y) with $x, y \in \mathbb{R}$. More formally, the xy-plane is the cartesian product between $\mathbb{R}$ and $\mathbb{R}$. The point (x, y) is just a pair of real valued numbers and we can write that $(x, y) \in \mathbb{R} \times \mathbb{R}$ or $(x, y) \in \mathbb{R}^2$ for short. In Figure 2.3, the region highlighted is the cartesian product between the positive real numbers which is often denoted as $\mathbb{R}^+$.

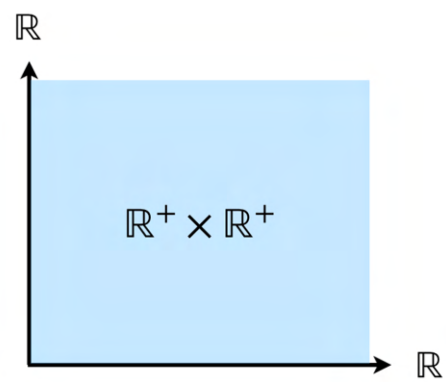


Figure 2.3: Example of the cartesian product between two sets

2.1.1 Equivalence Relations

Suppose that you are given two objects and told that these objects are equivalent, then this means that that they are the same in some sense. For example, the fraction 2/5 is equivalent to 4/10, the Spanish phrase "buenos dias" is equivalent to the English phrase "good morning", and a size 6 in women's shoes according to a United States size is equivalent to wearing a 36 in a European size. Similarly, there is also a notion of two objects being equivalent in a set and the objective of this section is to provide a precise mathematical definition of equivalence.

Definition 6. *Let X be a set and define $\sim$ to be a relation on X, then this relation is an equivalence relation if it satisfies the following criteria:*

1. ***Reflexive:*** $a \sim a$ *for all* $a \in X$.
2. ***Symmetric:*** *if* $a \sim b$, *then* $b \sim a$ *for any* $a, b \in X$.
3. ***Transitive:*** *if* $a \sim b$ *and* $b \sim c$, *then* $a \sim c$ *for any* $a, b, c \in X$.

One of the key steps in solving the 100 Rooms Problem is to define an equivalence relation on the set of real valued sequences, which is an exercise in the next chapter. For now, let's look at an example of how to prove that a relation is an equivalence relation and an example of a relation that is not an equivalence relation.

Example 3. *Let* $X = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \in \mathbb{N}\}$ *be the set that is illustrated in the left image in Figure 2.5. Now define the relation* $\sim$ *such that* $(x_1, y_1) \sim (x_2, y_2)$ *if* $x_1 = x_2$, *then this relation is an equivalence relation*

Proof.

1. **Reflexivity**: $(x, y) \sim (x, y)$ for any $(x, y) \in X$ because $x = x$.
2. **Symmetry**: Suppose that $(x_1, y_1) \sim (x_2, y_2)$ so $x_1 = x_2$, this implies that $x_2 = x_1$ so $(x_2, y_2) \sim (x_1, y_1)$.
3. **Transitivity**: Suppose that $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$, then $(x_1, y_1) \sim (x_3, y_3)$ because $x_1 = x_2 = x_3$.

$\square$

Example 4. *The relation "$\leq$" does not define an equivalence relation over the real numbers. This relation is reflexive because* $x \leq x$ *is a true statement for any* $x \in \mathbb{R}$. *Transitivity holds because* $x \leq y$ *and* $y \leq z$ *implies that* $x \leq z$. *However, this relation is not symmetric because* $3 \leq 5$ *but* $5 \leq 3$ *is not true.*

When we define an equivalence relation on a set, this gives us a natural way of finding a *partition* of the set. The word partition means to divide a set into distinct pieces such that every element in the set is contained in exactly one piece and none of the pieces overlap. For example, the United States can be partitioned into 50 states, a building can be partitioned into its levels, and a pizza can be partitioned into slices. In Figure 2.4, the image on the left shows a set X and the image on the right shows an example of a partition of X.

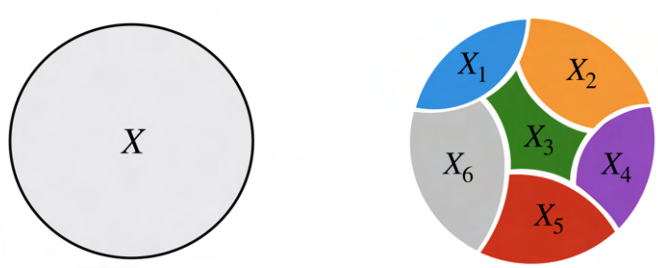


Figure 2.4: Example of a partition of a set

Definition 7. *Let X be a set and $\{X_i\}_{i \in \mathcal{I}}$ be a collection of subsets such that $X_i \subset X$ for all $i \in \mathcal{I}$ with $\mathcal{I}$ being the index set. Then $\{X_i\}_{i \in \mathcal{I}}$ is a partition of X if it satisfies the following criteria:*

1. *$X_i \neq \emptyset$ for all $i \in \mathcal{I}$*
2. *$X_i \cap X_j = \emptyset$ for all $i, j \in \mathcal{I}$*
3. *$X = \bigcup\limits_{i \in \mathcal{I}} X_i$.*

The first property in this definition means that none of the pieces in the partition can be empty. For example, you can't partition a pizza into slices and find a slice that doesn't contain any of the pizza. The second property means that none of the pieces can overlap and the last property means that you get the original set if you put all the pieces back together. Now the most important concept in this section is that an equivalence relation can be used to partition a set. Each piece in this partition is called an equivalence class because it consists of elements that are equivalent and only elements that are equivalent. Another way of saying this is that if $x \sim y$, this implies that x and y belong to the same equivalence class.

Theorem 1. *Let X be a set and $\sim$ be an equivalence relation on X, then this equivalence relation defines a partition of X into a set of equivalence classes.*

Example 5. *Let $X = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \in \mathbb{N}\}$ be the same set from Example 3 with the equivalence relation $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 = x_2$. Then the equivalence classes of this set are the different colored lines as shown in Figure 2.5.*

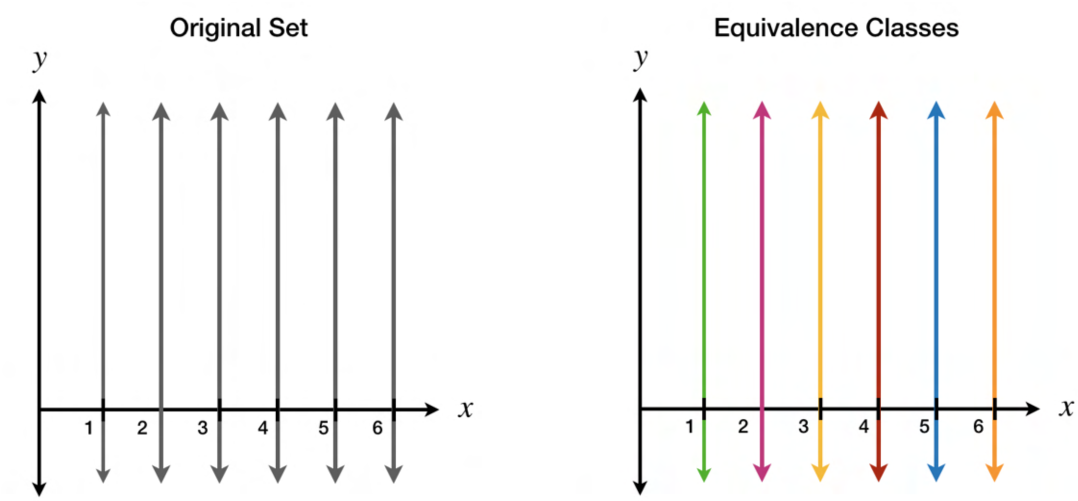


Figure 2.5

2.1.2 Mathematical Induction

There is a famous mathematician named Johann Gauss who is considered to be one of the greatest mathematicians of all time. He made many fundamental contributions to mathematics such as complex numbers, differentiating between different types of curvature on 3D shapes, and hyperbolic geometry which is mentioned in Section 2.3. There is a legend that as a child, he was asked to add together all of the numbers from 1 to 100. Instead of manually adding 100 hundred numbers together, he developed a formula that computes the sum of all the numbers from 1 to n where $n \in \mathbb{N}$.

Proposition 1. *Let $\mathcal{G}aus(n)$ and $\mathcal{S}um(n)$ be the functions defined by*

$$\mathcal{G}aus(n) = \frac{n(n+1)}{2} \qquad \textit{and} \qquad \mathcal{S}um(n) = \sum_{k=1}^{n} k,$$

then $\mathcal{G}aus(n) = \mathcal{S}um(n)$ for any $n \in \mathbb{N}$.

Its easy to check that Gauss' formula works when n is small, but how can we know that this formula works for all $n \in \mathbb{N}$? Instead of checking that his formula is true for every $n \in \mathbb{N}$, we can use a mathematical tool called induction to provide a *proof* that this formula always works. The main idea of an inductive argument is that we have a sequence of statements, namely $\mathcal{S}(1), \mathcal{S}(2), \ldots, \mathcal{S}(n), \ldots$, and we need to prove that these statements are true for any $n \in \mathbb{N}$. In our case, the statements we want to prove are true are the following

$$\mathcal{S}(n): \qquad \mathcal{G}aus(n) = \mathcal{S}um(n).$$

The first step is to verify that the first statement $\mathcal{S}(1)$ is true. Next, we assume that $S(n)$ is true for any $n \in \mathbb{N}$, then show that $S(n)$ being true implies that $S(n+1)$ is true. Sometimes induction is called a *domino argument* because since we have shown that $\mathcal{S}(1)$ is true and that $S(n)$ being true implies that $S(n+1)$ is true, this implies that $\mathcal{S}(2)$ is true which implies that $\mathcal{S}(3)$ is true and etc. Now let's use an inductive argument to show that Gauss' formula is true for any $n \in \mathbb{N}$.

Proposition 1. *Let $\mathcal{G}aus(n)$ and $\mathcal{S}um(n)$ be the functions defined by*

$$\mathcal{G}aus(n) = \frac{n(n+1)}{2} \qquad \textit{and} \qquad \mathcal{S}um(n) = \sum_{k=1}^{n} k,$$

then $\mathcal{G}aus(n) = \mathcal{S}um(n)$ for any $n \in \mathbb{N}$.

Proof.

1. **Base Case:**

$$\mathcal{G}aus(1) = \frac{1(1+1)}{2} = 1 \qquad \text{and} \qquad \mathcal{S}um(1) = \sum_{k=1}^{1} k = 1$$

   which implies that $\mathcal{S}(1)$ is a true .

2. **Inductive Hypothesis:** Assume that $\mathcal{S}(n)$ is true for any $n \in \mathbb{N}$.

3. **Inductive Step:**

$$\begin{aligned}
\mathcal{S}um(n+1) &= \sum_{k=1}^{n+1} k \\
&= n+1+\sum_{k=1}^{n} k \\
&= n+1+\frac{n(n+1)}{2} \qquad (2.1) \\
&= \frac{2(n+1)}{2}+\frac{n(n+1)}{2} \\
&= \frac{(n+1)(n+2)}{2} \\
&= \mathcal{G}aus(n+1)
\end{aligned}$$

where line (2.1) hold under the assumption that $\mathcal{S}(n)$ is true. □

Induction is a powerful tool that is used throughout these notes. Every inductive argument follows the basic outline used in the proof of Proposition 1 and a general outline is shown in Figure 2.6.

Let $\mathcal{S}(1), \mathcal{S}(2), \mathcal{S}(3), \ldots$ be any sequence of statements

1. Base Case:

Show that $\mathcal{S}(1)$ is true

2. Inductive Hypothesis

Assume that $\mathcal{S}(n)$ is true for any $n \in \mathbb{N}$

3. Hypothesis Step

Show that $\mathcal{S}(n+1)$ is true by using that $\mathcal{S}(n)$ is true

Figure 2.6: Outline of Inductive Argument

2.2 Cardinality

In this section, we introduce the notion of *cardinality* which is the size of a set and describe how to determine the cardinality. This is a simple task when a set is finite, but this problem becomes much more challenging and interesting when a set contains infinitely many elements.

For the sake of illustration, suppose that you are given the set $X = \{a, b, c\}$ and your task is to determine the cardinality of this set. A natural choice would be that the size of X is 3 because this set contains 3 elements which is the same answer that a mathematician would provide.

Let's consider a more interesting case, suppose that you are given the set of natural numbers $\mathbb{N}$ and asked what is the cardinality of $\mathbb{N}$? Perhaps the cardinality is infinite because this set contains infinitely many elements. This is a good answer, until your task is to determine the cardinality of the set of real numbers $\mathbb{R}$. In this case, it's still reasonable to say that the cardinality is infinite. But we can give a more precise answer by differentiating between different sizes of infinite sets. Thus, the goal of this section is to explain this difference and describe some surprising properties of these sets.

2.2.1 Finite Sets

When a set is finite, the cardinality is determined by *counting* the number of elements in the set. For example, suppose that the set $X = \{x_1, x_2, \ldots, x_{k-1}, x_k\}$ contains k elements, then the cardinality of X is k which is denoted by $|X| = k$. The most important concept in this section is that when two finite sets have the same cardinality, then there is a correspondence between these sets. The main idea is that given two sets A and B with the same cardinality, every element $a \in A$ *corresponds* to exactly one element $b \in B$. Moreover, we can define a function that defines the correspondence between the two sets, but this mapping must satisfy two conditions.

Definition 8. *Let $f : A \to B$ be a function that defines a mapping between two sets A and B. This function is one-to-one if it maps each element in A to unique element in B and onto if for any $b \in B$ there is an $a \in A$ such that $f(a) = b$.*

Definition 9. *Let $f : A \to B$ be a function between two sets A and B, then f is a bijection if it is both one-to-one and onto.*

A few quick note on notation, the phrase "$f : A \to B$" means that f is a function that takes an element $a \in A$ and maps this element to some $b \in B$ i.e $f(a) = b$. For example, you can think of a gum ball machine as a function, where the input is a quarter and the output is a gum ball. This function being one-to-one means that whenever you put a quarter into the machine, you always get exactly one gum ball out. This machine being onto means that with enough quarters, I can get any gum ball out of the machine.

Definition 10. *A one-to-one correspondence is a relationship between two sets A and B such that there exists a bijective function $f : A \to B$.*

Another quick note on notation is that the term *one-to-one correspondence* is a misnomer because the function must be both one-to-one *and* onto. In order to show that two sets have a one-to-one correspondence, we need to first define a function between the two sets. Then show that this function is a bijection. Next, lets look at an example of two sets A and B that have a one-to-one correspondence.

Example 6. *Let $A = \{2, 7, 3, 8, 1\}$ and $B = \{4, 9, 5, 6, 0\}$, then define $f : A \to B$ by $f(2) = 9, f(7) = 5, f(3) = 6, f(8) = 4,$ and $f(1) = 0$ which is illustrated in Figure 2.7. Then the function f is a bijection, so it defines a one-to-one correspondence between the sets A and B.*

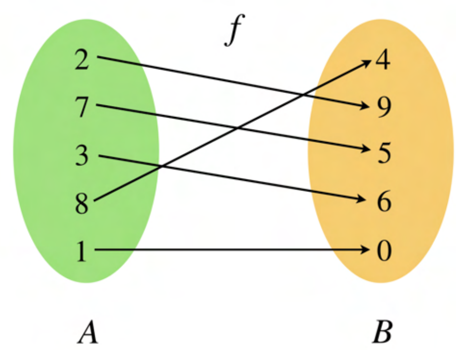


Figure 2.7: One-to-one correspondence

An important point is that in order for two sets to have a one-to-one correspondence, the sets must have the exact same cardinality. Otherwise, if two sets do not have the same cardinality, its impossible to find a bijective function between the two sets. In this case, any function that we attempt to construct will fail to be either one-to-one or onto. Next, let's look at some examples of a function failing to define a one-to-one correspondence.

Example 7. *In Figure 2.8, the function fails to be onto because there is no element in A that maps onto $1 \in B$. In Figure 2.9, the function fails to be one-to-one because the elements $2, 5 \in A$ both map to $4 \in B$ i.e $f(2) = f(5) = 4$.*

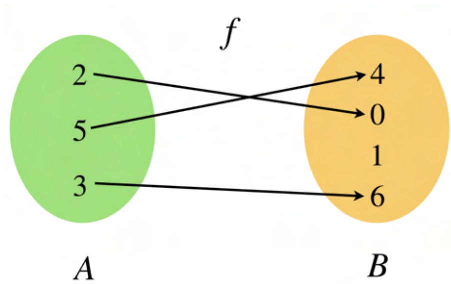


Figure 2.8

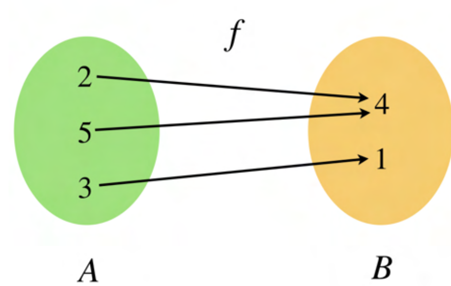


Figure 2.9

2.2.2 Countably Infinite Sets

The more challenging case of determining cardinality is when a set contains infinitely many elements. In this case, the cardinality is either countably infinite or uncountably infinite with the latter being the larger of the two. The difference between these cardinalities is that some infinite sets can be counted, whereas others cannot. When we say that a set can be *counted*, this means that all of the element can be listed one by one. For example, the set shown in Figure 2.10 is countable because we can create a list of all its elements, the technical name for this list is an *enumeration*. One key observation of the enumeration in Figure 2.10 is that there is a one-to-one correspondence between the set shown and the set $\{1, 2, 3, 4\}$, which are called the indices of the list. The definition of a countably infinite set is a generalization of what it means to be countable. A set is countably infinite if all of the elements can be listed one by one in an enumeration that is indexed by the natural numbers.

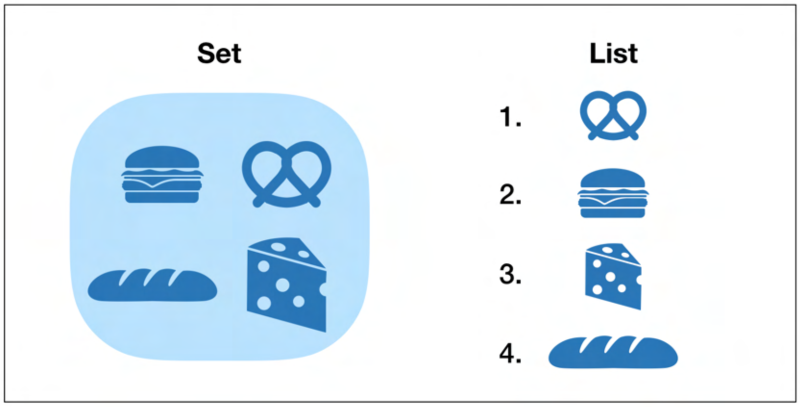


Figure 2.10: Example of an enumeration of a set

Definition 11. *A set X is countably infinite if there exists a one-to-one correspondence between X and the natural numbers $\mathbb{N}$.*

The definition of a countably infinite set relies on the concept of a one-to-one correspondence which was defined in Definition 10 from the previous section. In terms of this definition, a set X being countably infinite means that there exists a function $f : \mathbb{N} \to X$ or $f : X \to \mathbb{N}$ that defines a bijection between the two sets. This definition uses a bijection to formalize what it means to make a list of all the elements. For example, suppose our bijection is $f : \mathbb{N} \to X$, then the first element on our list is $f(1)$, the second is $f(2)$, and etc. One practice thats very common is that if X is a countably infinite set, then we can find an enumeration such that $X = \{x_1, x_2, x_3, \ldots\}$ where $x_1 = f(1)$, $x_2 = f(2)$, and $x_3 = f(3)$. This list is very similar to the one in Figure 2.10, but the difference is that its infinity long.

Next, lets look at an example of how to prove that a set is countably infinite. In order to prove this, we must find a function between $\mathbb{N}$ and X that is both one-to-one and onto, hence a bijection. In the next proposition, we use another type of mathematical argument called *proof by contradiction*. The setting for this type of proof is that we have some statement we need to prove is true. The argument is to assume that this statement is false and show that this assumption leads to a contradiction. Thus, the original statement that we assumed to be false, must be true because otherwise this leads to a contradiction.

Proposition 2. *Let $E = \{x \in \mathbb{N} : x \text{ is even }\}$ be the set of all even numbers, then E is countably infinite.*

Proof. Define the function $f : \mathbb{N} \to E$ by $f(x) = 2x$ for any $x \in \mathbb{N}$. Notice the pattern that $f(1) = 2, f(2) = 4, f(3) = 6$, and etc, which is illustrated in Figure 2.11.

1. **One-to-One**: By contradiction, suppose that f is not one-to-one so there exists some $x_1, x_2 \in \mathbb{N}$ such that $f(x_1) = f(x_2)$ and $x_1 \neq x_2$. But then $2x_1 = 2x_2$ by the definition of f and this implies that $x_1 = x_2$. Since our assumption that f is not one-to-one leads to a contradiction, then f must be one-to-one.

2. **Onto**: Choose any $x \in E$ and since x is even, there must exist some $n \in \mathbb{N}$ such that $x = 2n$. Now using the definition of f, we get that $f(n) = 2n = x$ which implies that f is onto.

$\square$

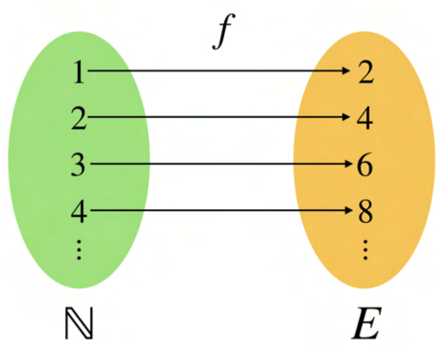


Figure 2.11: One-to-one Correspondence

This result may seem a bit paradoxical because $E \subset \mathbb{N}$ and $|E| = |\mathbb{N}|$, meaning that there are just as many even numbers as there are natural numbers. Perhaps this seems counterintuitive because this would be impossible when the sets are finite. We will revisit this proposition at the end of the next section and provide a second proof of it. Perhaps a more shocking result is that there are just as many rational numbers as there are natural numbers.

Theorem 2. *The cardinality of the set of rational numbers $\mathbb{Q}$ is countably infinite.*

Proof. Every rational number $x \in \mathbb{Q}$ can be written as $x = p/q$ with $p, q \in \mathbb{N}$. We can generate an enumeration of the rationals by using an infinitely long and wide grid as shown in Figure 2.12. Let G be this grid such that $G(p, q) = p/q$ refers to the entry in row p and column q, which means that every rational number can be found in this grid. We can define a bijection $f : \mathbb{N} \to \mathbb{Q}$ by looping through the grid as illustrated in the rightmost image in Figure 2.12, so that the function f is defined by $f(1) = 1, f(2) = 1/2, f(3) = 2/1$ and etc. Then this function is a bijection between the rational and natural numbers. □

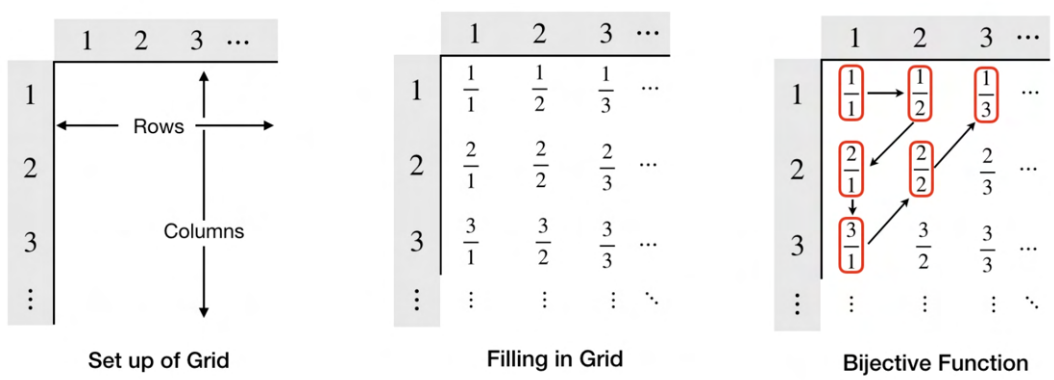


Figure 2.12

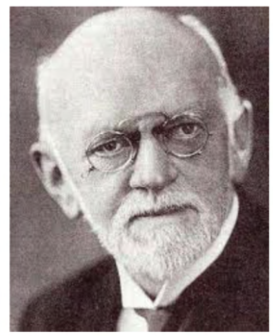
This famous argument was first made in the late 1800s by George Cantor who is the founding father of set theory and revolutionized our understanding of infinity. Prior to his work, mathematicians assumed that a set is either finite or infinite and made no distinction between different sizes of infinity. In fact, the concept of infinity was considered to be a philosophical topic, where philosophers argued over ideas such as whether an actual infinity exists.

Many of Cantor's theories were initially considered radical and mostly rejected. One of his former professor's named Leopold Kronecker fiercely opposed Cantor's ideas and is known for saying "God made the integers; all else is the work of man." He believed that only mathematical objects created in a finite way could have meaning. It took decades before Cantor's ideas were accepted and the mathematical community further developed his theories.

The turning point was at the brink of the 20th century at an international math conference held in Paris. A mathematician named David Hilbert presented a list of 23 unsolved problems that he considered to be the most important problems of the century. The first problem on the list was called the Continuum Hypothesis which was first posed by Cantor, but he was unable to work out a solution. We'll state and discuss this problem in Section 2.2.4 after we cover uncountably infinite sets. To conclude this section, although Cantor faced much opposition his work eventually became accepted and has been very influential on modern mathematics.

George Cantor

Leopold Kronecker



David Hilbert

2.2.3 Hilbert Hotel

The *Hilbert Hotel* is a famous thought experiment that was originally created by Hilbert in 1924. The purpose of his thought experiment is to illustrates some paradoxical properties of countably infinite sets.

Imagine that you are the manager of the Hilbert Hotel which has a countably infinite number of rooms. This means that the rooms can be labelled by the natural numbers and enumerated as Room 1, Room 2, Room 3, and etc. The hotel is completely full and suppose that a new guest arrives. As the manager, you must design a room assignment scheme to assign all of the current and the newly arrived guest to a room in the hotel.

The paradoxical result is that it's possible to accommodate all of the guests despite the hotel being completely full. Now if you were the manger of the *Kronecker* Hotel, then this task would be impossible. However, the Hilbert hotel has infinitely many rooms and we can accommodate the new guest by shifting every current guest to the next room over. Now the new room assignment is that the current guest in Room 1 moves to Room 2, the guest in Room 2 moves to Room 3, and etc, so now the new guest can stay in Room 1.

We can even generalize this room assignment scheme to the case when n guests arrive. Perhaps an even more shocking result is that we can accommodate a countably infinite number of newly arrived guests when the Hilbert Hotel is completely full.

Proposition 3. *Suppose that a countably infinite number of new guests arrive at the Hilbert Hotel, then it is possible to assign every newly arrived and current guest to a room.*

Proof. To accommodate the new guests, move the current guest in Room 1 to Room 2, the guest in Room 2 to Room 4, the guest in Room 3 to Room 6, and etc. The room reassignment for the current guests is that the guest in Room n moves to Room $2n$. As the current guests move to the even numbered rooms, we can assign the newly arrived guests to the odd numbered rooms by using Exercise 14 in Section 2.4. □

This thought experiment gives us an alternative way of thinking about countably infinite sets and another way of proving that a set is countable. Suppose that every room in the Hilbert Hotel is empty, then a set is countably infinite if every element in the set can be assigned to a unique room and the elements occupy every room of the hotel. Next, let's prove that the set of even numbers is countably infinite by making an argument using the Hilbert Hotel.

Example 8. *Suppose that every room in the Hilbert Hotel is empty and a group of guests arrive and assume that they are labelled by the set of even numbers, so the guests are Guest 2, Guest 4, Guest 6, etc. Then the room assignment is that Guest n stays in Room $n/2$, which is illustrated in Figure 2.13. Notice that this one-to-one correspondence between guests and rooms is identical to the one-to-one correspondence used in Proposition 1.*

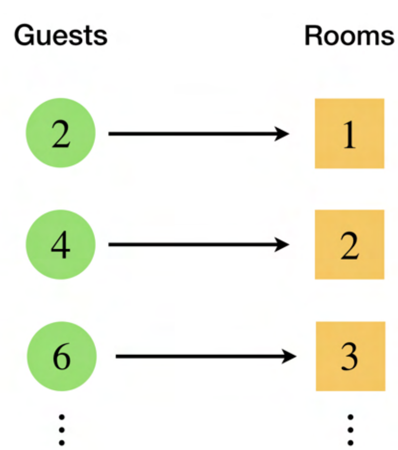


Figure 2.13

2.2.4 Uncountably Infinite Sets

One of Cantor's most remarkable contributions to mathematics is that he was the first to prove that there are different sizes of infinite sets, namely countably infinite and uncountably infinite. An uncountably infinite set is larger than a countably infinite set in the sense that it is impossible to enumerate all of its elements.

Definition 12. *The cardinality of a set is uncountably infinite if the set contains too many elements to define a one-to-one correspondence with the natural numbers.*

When Cantor gave the first proof of a set being uncountable, he used a proof by contradiction where he assumed a set can be enumerated but then found an element that was not contained in the enumeration. This technique can be applied to many different examples of uncountable sets with the simplest being the set of all binary sequences $\mathcal{B}$. To be clear, a binary sequence is an infinitely long list of 0s and 1s. For example, $x = (0, 1, 1, 1, 0, \ldots)$ is a binary sequence so $x \in \mathcal{B}$ and an example of an enumeration of $\mathcal{B}$ is shown in Figure 2.14.

Proposition 4. *The cardinality of the set of binary sequences $\mathcal{B}$ is uncountable.*

Proof. By contradiction, assume that the cardinality of $\mathcal{B}$ is countable. Under this assumption, this set can be enumerated so that $\mathcal{B} = \{x_1, x_2, x_3, x_4, \ldots\}$. Let $x_n(k)$ denote the k-th term in the n-th sequence of the enumeration. For example, $x_2(4) = 1$ according to the enumeration shown in Figure 2.14. Then define the sequence $x^\star = (1 - x_1(1), 1 - x_2(2), 1 - x_3(3), 1 - x_4(4), \ldots)$ which cannot be contained in the enumeration because $x^\star$ differs from each element by at least one term. The existence of this binary sequence contradicts our assumption that $\mathcal{B}$ is countable, so $\mathcal{B}$ must be uncountable. $\square$

$$\begin{aligned}
x_1 &= (1, 0, 1, 0, 0, \ldots)\\
x_2 &= (0, 1, 1, 1, 1, \ldots)\\
x_3 &= (1, 0, 0, 1, 0, \ldots)\\
x_4 &= (0, 0, 0, 1, 1, \ldots)\\
x_5 &= (1, 1, 1, 0, 0, \ldots)\\
&\ \vdots
\end{aligned}$$

Enumeration

$$\begin{aligned}
x_1 &= (\boxed{1}, 0, 1, 0, 0, \ldots)\\
x_2 &= (0, \boxed{1}, 1, 1, 1, \ldots)\\
x_3 &= (1, 0, \boxed{0}, 1, 0, \ldots)\\
x_4 &= (0, 0, 0, \boxed{1}, 1, \ldots)\\
x_5 &= (1, 1, 1, 0, \boxed{0}, \ldots)\\
&\ \vdots
\end{aligned}$$

Diagonalization

Figure 2.14

The main idea of this proof is to construct a sequence $x^\star$ that is not contained in the enumeration by forcing this sequence to differ from every sequence in the enumeration by at least one term. The first term in the sequence $x^\star$ is defined by $x^\star(1) = 1 - x_1(1) = 0$ which enforces that $x^\star \neq x_1$, then continue this pattern for each term of $x^\star$. If we apply this argument to the enumeration in Figure 2.14, we obtain the sequence $x^\star = (0, 0, 1, 0, 1, \ldots)$. The technique of constructing a sequence from the diagonal terms is called *Cantor's diagonalization argument* and this argument can be generalized to prove that the set of real numbers is also uncountably infinite.

Lemma 1. *Let U and V be two sets. If V is uncountably infinite and $V \subset U$, then U must also be uncountably infinite.*

Theorem 3. *The cardinality of the set of real numbers is uncountably infinite.*

Proof. Suppose that the set of real numbers contained in the interval $[0,1] \subset \mathbb{R}$ is countable so there exists an enumeration such that $\mathbb{R} = \{x_1, x_2, x_3, x_4, \ldots\}$. Define $x_n(k)$ to be the k-th number in the decimal expansion of the n-th real number in the enumeration. Then let $x^\star = 0.d_1d_2d_3d_4\cdots$ where $d_n \sim \{0,\ldots,9\}\backslash\{x_n(n)\}$. This implies that $x^\star$ cannot be contained in the enumeration because it differs from each element by at least one term. Thus, the interval [0,1] is uncountable and since $[0,1] \subset \mathbb{R}$, Lemma 1 implies that the real numbers must also be uncountably infinite. $\square$

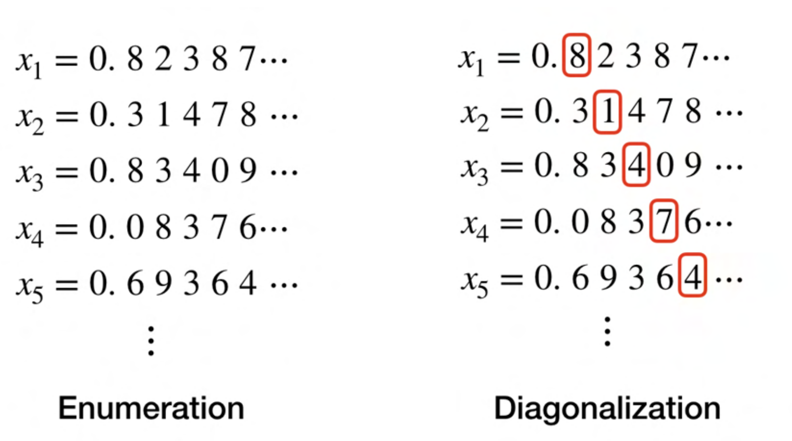
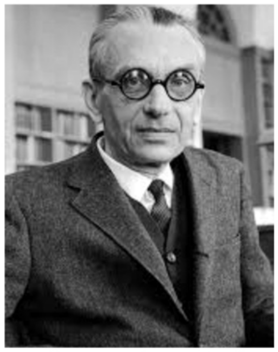


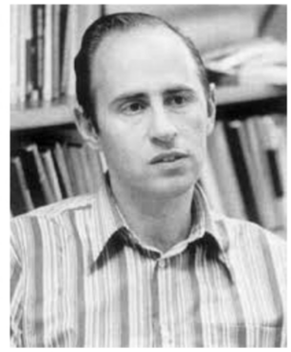
Figure 2.15

Collary 1. *The set of irrational numbers is uncountable.*

In a previous section, we mentioned a problem called the Continuum Hypothesis which was first posed by Cantor in the late 1800s and took about 100 years to partially resolve. The problem is to determine whether there exists a set whose cardinality is greater than the natural numbers but less than the reals. Cantor believed that this statement is false, but was unable to work out a proof. In the early 1900s, this was the first problem on Hilbert's list of the 23 most important problems of the century. The first break through in this problem was made in the 1940s by an Austrian mathematician named Kurt Gödel. An interesting but unrelated fact is that he spent many years working at Princeton and he was good friend of Einstein. Anyways, he was able to show that if the continuum hypothesis is true and if it were added a framework of set theory called Zermelo-Fraenkel set theory, then its addition would not contradict this type of set theory. But he was unable to determine whether this statement is true or false. The next breakthrough was in the 1980s due to an American mathematician named Paul Cohen who was a professor at Stanford for many years. He was able to show that no contradiction would arise if the continuum hypothesis is false. If we combine Gödel's result with Cohen's, then the statement is undecidable and its validity is dependent on the framework of set theory.



Kurt Godel



Paul Cohen

2.3 Axiom of Choice

The statement of the axiom of choice is quite simple, but controversial because this it can be used to prove some very strange results. An axiom is a mathematical statement that is assumed to be true, as opposed to a statement that is proven or disproven. For example, the topics taught in a high school geometry class are drawn from Euclidean geometry which is built on the following axioms:

1. A straight line segment can be drawn joining any two points.
2. Any straight line segment can be extended indefinitely as a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one point as center.
4. All right angles are congruent.

Under the assumption of these four statements being true, it is possible to prove classical results such as the Pythagorean Theorem and the Triangle Sum Theorem. Moreover, everything taught in a high school geometry class relies on the assumption of these four axioms being true. However, there was one geometry problem called the *parallel postulate* which stumped mathematicians for two thousand years. This problem was first posed by the ancient Greek mathematician Euclid and it states that if two parallel lines are extended indefinitely, then these lines never intersect. This was a well known problem and there were many attempts to solve it. Even Gauss worked on the problem, but was also unable to work out a solution. This problem remained unsolved until the early 1800s when two mathematicians named Janos Bolyai and Nicolai Lobachevsky independently realized that this statement can be neither proven nor disproven. This means that the parallel postulate is an axiomatic statement, where you get choose to whether the statement is true or false. This discovery led to two different branches of geometry called Euclidean which assume this postulate is true, whereas hyperbolic geometry assumes that this statement is false. In fact, the image on the title page of these notes is a drawing by M.C. Escher that illustrate an infinite tessellation on a hyperbolic disk.

Stefan Banach

Alfred Tarski

The axiom of choice is similar to the parallel postulate because both of these statements are axioms that must be either accepted or rejected. This axiom was implicitly used for many years, but there was a disagreement over whether it is a valid statement. Although there are several fundamental theorems that rely on the axiom of choice, there are some very strange results that can be proven by using this axiom. The most infamous example is the *Banach-Tarski Paradox* which was the work of two Polish mathematicians named Stefan Banach and Alfred Tarski. They were able to prove that given a solid 3-dimensional sphere, it is possible to cut the sphere into five pieces and reassemble these pieces in a way that creates to spheres with the same radius as the original sphere. In addition, they proved a stronger result that its possible to cut a sphere the size of a pea into five pieces and reassemble these pieces into a sphere that's as big as the sun. These are some of the most remarkable and shocking results of the 20th century because they completely violate our intuition. Some criticize this result because the disassembled pieces are so irregular that they have its impossible to measure their volume. So it's physically impossible to take something like a basketball, cut it up into pieces such as in the Banach-Tarski Paradox, and then put them back together to get two basketballs. However, if we choose to reject the axiom of choice, then we can proven even stranger results such as the existence of an uncountably infinite subset of real numbers without a countably infinite subset.

Figure 2.16: Banach-Tarski Paradox

The One Hundred Rooms Problem is another example of a paradoxical result that can be proven by using the axiom of choice. In its simplest form, this axiom guarantees that given any set, we can always *choose* exactly one element from the set. This single element is referred to as the representative in the context of the axiom of choice. For example, suppose that you have a bag of marbles, you could easily choose one marble from the bag. In this scenario, there's no formula for how to choose the marble or what marble will be chosen. But its a natural assumption that its possible to choose some representative. The axiom of choice in its full form is that given a collection of sets, then its possible to choose exactly one element from each set. In the case, the scenario where we choose representatives from a collection of bags is illustrated in Figure 2.17.

Definition 13. *Let X be a set, then a choice function $f : X \to X$ is a set function that returns exactly one element from the set.*

Axiom of Choice. *For any collection of sets $X = \{X_i\}_{i \in \mathcal{I}}$, there exists a family of representatives $\mathcal{F} = \{f_i(X_i)\}_{i \in \mathcal{I}}$ where f_i is a choice function and $\mathcal{I}$ is some index set.*

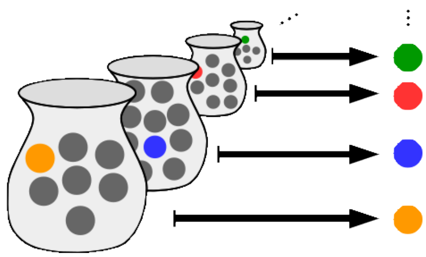


Figure 2.17: Axiom of Choice

The notion of choosing representatives from a set is formalized by the use of a choice function. This function takes a set as the input and returns exactly one element from this set. The axiom of choice guarantees the existence of a choice function, but does not specify how to construct this function for any given set. Moreover, the axiom guarantees that for any collection of sets, there exists a choice function for every set in the collection. Next, let's look at some simple examples where we can define a choice function.

Example 9. *Let $\mathcal{P}(\mathbb{N})$ be the power set of the natural numbers and define a choice function over this collection of sets. Given that $x \geq 0$ for all $x \in \mathbb{N}$, then let $f(X) = \min X$ for any $X \in \mathcal{P}(\mathbb{N})$.*

The main idea in this example is that any subset of the natural numbers has a smallest element because $x > 0$ for all $x \in \mathbb{N}$. Using this property of these sets, we can just choose the smallest element in each set to be the representative. To illustrate why this axiom is controversial, imagine that our collection of sets is the power set of the reals i.e. $\mathcal{P}(\mathbb{R})$. The sets in this collection can be so irregular that its not clear how to even start thinking about defining a choice functions. Although there is a history of controversy behind the axiom of choice, nearly every mathematician accepts this axiom now.

2.4 Exercises

Set Theory

1. Let $A = \{1, 7, 5, 0, 2, 4\}$, $B = \{1, 8, 9, 5, 2, 4\}$, and $C = \{5, 7, 3, 9, 2\}$, now perform the following set operations.

   (a) $A \cup B$

(b) $A \cap C$

(c) $(A \cup B) \backslash (A \cap B)$

2. Let $A = \{1, 7, 2, 3, 10\}$, $B = \{3, 5, 7, 2, 9\}$, and $C = \{1, 7, 4, 8, 2\}$, show that the following identity is true
$$A \cap \big(B \cup C\big) = (A \cap B) \cup (A \cap C)$$
by computing each side of the equality and following the order operations.

3. List all of the elements in the set $X = \{x \in \mathbb{N} : x < 10\}$.

4. Let $X = \{a, b, c, d\}$ and compute $\mathcal{P}(X)$.

5. Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$, now list all of the elements of the set $A \times B$. Just for fun, can you think of a popular game where you might use the elements of $A \times B$.

Equivalence Relations

6. Let X be the set of people with an account on Facebook and define the relation that $x \sim y$ if x is friends with y. Does this relation define an equivalence relation and explain why or why not?

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and define the relation $x_1 \sim x_2$ if $f(x_1) = f(x_2)$. Show that this defines an equivalence relation and what do the equivalence classes look like when $f(x) = x^2$ and $f(x) = \sin(x)$.

8. Let X be the set of all real-valued functions and define the relation $\sim$ on X such that $f \sim g$ if there exists some constant $k \in \mathbb{R}$ such that $f(x) = kg(x)$ for all $x \in \mathbb{R}$.

9. Let $X = \{a, b, c, d\}$ be a set.
   (a) Determine the power set $\mathcal{P}(X)$.
   (b) Show that the relation $U \sim V$ when $|U| = |V|$ for any $U, V \in \mathcal{P}(X)$ is an equivalence relation.
   (c) Find the equivalence classes of $\mathcal{P}(X)$ corresponding to the equivalence relation.

10. (Challenge) Let $\mathcal{P}(\mathbb{N})$ be the power set of the naturals and define the relation $A \sim B$ if there exists a bijection between the two sets. Show that this relation defines an equivalence relation on $\mathcal{P}(\mathbb{N})$.

Induction

11. Prove that $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all $n \in \mathbb{N}$.

12. Prove that $3 + 11 + \cdots + (8n - 5) = 4n^2 - n$ for all $n \in \mathbb{N}$.

Cardinality

13. Let $\mathcal{B}(n)$ be the set of all binary sequences of length n, which is an ordered list of 0s and 1s. For example, the sequence $x_1 = (0, 0, 1, 1)$ is a binary sequence with $x_1 \in \mathcal{B}(4)$. Determine the cardinality of the set of all binary sequences of length n?

14. Show that the cardinality of the set of odd numbers is countably infinite.

15. (Challenge Problem) Show that the cardinality of the union of n countably infinite sets is countably infinite. (Hint: use induction)

16. Show that the cardinality of $\mathbb{Z}$ is countably infinite.

17. Suppose that A and B are countably infinite sets.

    (a) Show that $A \cap B$ is countably infinite.

    (b) Show that $A \cup B$ is countably infinite.

    (c) Show that $A \times B$ is countably infinite.

18. Prove that the set of irrational numbers is uncountably infinite.

19. (Challenge Problem) Suppose that X is a set such that $|X| = n$, find a formula for the cardinality of the power set of X. (Hint: use induction)

Hilbert Hotel

20. Suppose that n new guests arrive at the Hilbert Hotel. Design a room assignment scheme to accommodate the current and newly arrived guests.

21. (Challenge Problem) Suppose that n carriages arrive with each containing a countably infinite number of guests. Design a room assignment scheme to accommodate the current and newly arrived guests.

22. (Challenge Problem) Further generalize Exercise 21 to the case when a countably infinite number of carriages with each containing a countably infinite number of guests arrive. (Hint: problem 10 in Section 3.4 may be helpful.)

Axiom of Choice

23. Let $X = \{$Rhode Island, Massachusetts, Connecticut, Vermont, Maine, New Hampshire$\}$ be the following collection of states and assume that each set in X contains all of the cities in that particular state. Define a choice function on X.

24. Let X be the set of all intervals defined over the reals with rational end points. This means that every element $x \in X$ can be write as $x = [a, b]$ with $a, b \in \mathbb{Q}$. Define a choice function on X.

Chapter 3

Real Analysis

Figure 3.1: New York City

3.1 Metric Spaces

Suppose that you are asked "what is the distance between the Empire State Building and the Brooklyn Bridge" as illustrated in Figure 3.1. Perhaps the distance is the length of the straight line that connects the two locations. But if someone wanted to walk from the Empire State Building to the Brooklyn Bridge, then this straight line distance would be very inaccurate because a walker is confined to the streets. In this case, the distance should be length of the shortest path along streets that connect the two points. Either distance is reasonable, but the answer to this question is dependent on how you measure distance. The goal of this section is to introduce a function called a *metric* that measures the distance between two points in a set. For example, suppose that our set is every point of interest in New York City and we have some metric defined on this set. Then if we input two locations in New York City into the metric, the output is the distance between these points.

Definition 1. *Let X be a set, then a metric on X is a function $d : X \times X \to \mathbb{R}^+$ that satisfies the following for any $x, y, z \in X$:*

1. ***Non-negative:*** $d(x, y) \geq 0$ *and d(x,y)=0 if and only if* $x = y$
2. ***Symmetric:*** *d(x,y)=d(y,x)*
3. ***Triangle Inequality:*** $d(x, z) \leq d(x, y) + d(y, z)$

The first two conditions in the definition of a metric are quite intuitive: distance can't be negative, there are no two distinct points with zero distance between them, and the distance from x to y is the same as the distance from y to x. The last property may not seem intuitive at first glance, but is immediately clear in an illustration. Suppose you choose any two points $x, z \in X$, then compute the distance between these points. Now choose any third point $y \in X$ and imagine walking from x to y then y to z, this distance must be either the same or bigger than walking from x to z as illustrated in Figure 3.2. Here's another way of thinking about the Triangle Inequality, suppose you are at the Empire State Building and you decide to make a stop in Manhattan before going to the Brooklyn Bridge. Then this stop in the middle may not affect the length of the path or could make the trip longer. But it wouldn't make sense if stopping at an intermediary point made the trip shorter.

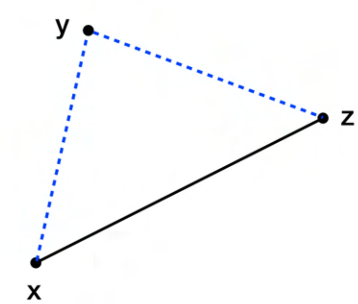


Figure 3.2: Triangle Inequality

In our example of defining distance in New York City, the blue and black dotted lines shown in Figure 3.1 correspond to different metrics. Suppose that the Empire State Building is at the point (x_1, y_1) and the Brooklyn Bridge is at (x_2, y_2), then the distance shown by the blue line can be computed from the metric $d((x_1, y_1), (x_1, y_1)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ which is called the Euclidean metric. The distance shown by the black line can be computed by $d((x_1, y_1), (x_2, y_2)) = |x_1 - y_1| + |x_2 - y_2|$ which is called the Manhattan metric or taxi cab metric. Next, let's look at another example of a metic and prove that it satisfies each of the three conditions in Definition 1.

Lemma 1. $|x + y| \leq |x| + |y|$ *for any* $x, y \in \mathbb{R}$.

Example 1. *Let* $X = \{x \in \mathbb{R} : x > 0\}$ *and define* $d(x, y) = |\log(x/y)|$ *for any* $x, y \in \mathbb{R}$ *with* $x, y > 0$*, then d is a metric.*

Proof.

1. It is clear that d is non-negative. First, $d(x, x) = \log(x/x) = \log(1) = 0$ for any $x \in X$. If $d(x, y) = 0$, this is only true when $x/y = 1$ which implies that $x = y$.

2. Symmetry holds because for any $x, y \in X$

$$\begin{aligned} d(x,y) &= |\log(x/y)| \\ &= |\log(x) - \log(y)| \\ &= |\log(y) - \log(x)| \\ &= |\log(y/x)| \\ &= d(y,x). \end{aligned}$$

3. Choose any $x, y, z \in \mathbb{R}$, then

$$\begin{aligned} d(x,z) &= |\log(x/z)| \\ &= |\log(x) - \log(z)| \\ &= |\log(x) - \log(y) + \log(y) - \log(z)| \\ &\leq |\log(x) - \log(y)| + |\log(y) - \log(z)| \qquad (3.1) \\ &= |\log(x/y)| + |\log(y/z)| \\ &= d(x,y) + d(y,z). \end{aligned}$$

where line (3.1) holds by Lemma 1. $\square$

We can define a metric on any type of set, even sets that may not have an underlying geometric structure. For example, we will define a metric on the set of all real valued sequences in order to solve the 100 Rooms Problem.

3.2 Sequences

The remainder of this chapter focuses on providing an elementary introduction to sequences and convergence, which are central to the study of real analysis. The mathematical tools introduced in this section are crucial to solving the 100 Rooms Problem. One of the key steps is to realize that boxes in each room are labelled by the natural numbers so they form a sequence of real numbers. Moreover, this ordering is identical in every single room, so we can use tools from real analysis to analyze this sequence and solve the problem.

Definition 2. *Let X be a set, then a sequence $x = \{x_n\}_{n=1}^{\infty}$ is an enumerated collection of objects such that $x = (x_1, x_2, x_3, \ldots)$ with $x_i \in X$ for all $i \in \mathbb{N}$.*

The key property of a sequence is that the order of the elements matters. This differs from a set because the elements are unordered in a set. Another difference is that the elements in a set cannot be repeated, whereas an element in a sequence may be repeated even infinitely many times. Next, let's look at some examples of famous sequences.

Example 2. *There is an infinitely long sequence of prime numbers.*

Proof. By contradiction, suppose there are only finitely many primes and let $x = (x_1, \ldots, x_n)$ be the sequence of all primes. But then $x^\star = x_1 \cdot x_2 \cdot x_3 \cdots x_{n-1} \cdot x_n - 1$ cannot be divisible by any prime number, which implies that $x^\star$ is prime but not contained in the sequence x. This is a contradiction, so there must be infinitely many primes $\square$

Example 3. *The Fibanocci sequence is defined by initializing $x_1 = 0$ and $x_2 = 1$, then generating the rest of the sequence by $x_n = x_{n-1} + x_{n-2}$.*

The first few terms of the Fibanocci sequence are $x = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots)$ which is illustrated in Figure 3.3. This sequence is very peculiar because there are many examples of these numbers being found in nature. For example, if you partition the flower as shown in the left image in Figure 3.3 and count the number of petals in each region, then this sequence of numbers is the Fibanocci sequence. There are many of examples of this sequence being found in nature such as in pine cones, sea shells, and spiral galaxies. Another surprising feature of the Fibonacci sequence is that if you compute the ratio between consecutive terms i.e. x_{n+1}/x_n, then this ratio approximates the golden ratio which shows up in art and architecture.

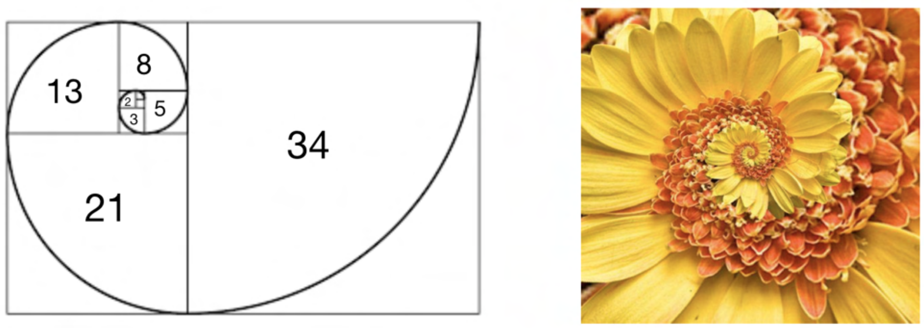


Figure 3.3: Fibonacci spiral and flower spiral

The next mathematical tool we need to solve the 100 Rooms Problem is to construct a subsequence from a sequence. The main idea is that given a sequence, we can extract some of its elements to create a new sequence which is called a subsequence. The only rule is that the elements in the subsequence must follow the same ordering as the original sequence.

Definition 3. *A subsequence of a sequence* $\{x_n\}_{n=1}^{\infty}$ *is formed by deleting elements of* x_n *to produce a new sequence* $\{x_{n_k}\}_{k=1}^{\infty}$. *A subsequence is indexed by the increasing sequence* $\{n_k\}_{k=1}^{\infty}$ *where each* n_k *is the index of the corresponding term in the original sequence.*

A subsequence may have a finite or infinite number of terms, but we will work with infinitely long subsequences to solve the 100 Rooms Problem. Next, let's look at some examples of constructing a subsequence from a sequence.

Example 4. *Let* $x = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots)$ *be a sequence and partition this sequence into two subsequences* y_1 *and* y_2. *Let* $y_1 = (1, \frac{1}{3}, \frac{1}{5}, \ldots)$ *which is indexed by* $n_k = 2k$ *for* $k \in \mathbb{N}$ *and let* $y_2 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots)$ *which is indexed by* $m_k = 2k + 1$ *for* $k \in \mathbb{N}$.

3.3 Convergence of a Sequence

The last mathematical tool we need to understand is what it means for a sequence to converge. The concept of convergence ties together metric spaces and sequences because a sequence converging to a point means that the distance between the terms in the sequence and the point become infinitely small.

To illustrate this concept, let's look at a famous problem called *Zeno Paradox* which was created by ancient Greek mathematicians. Suppose that Zeno would like to walk the entire length of path that is 1 km long. Before Zeno can reach the end of the path, he must walk half the distance so that he is 0.5km from the end. Again, he must walk half the distance between the 0.5km point

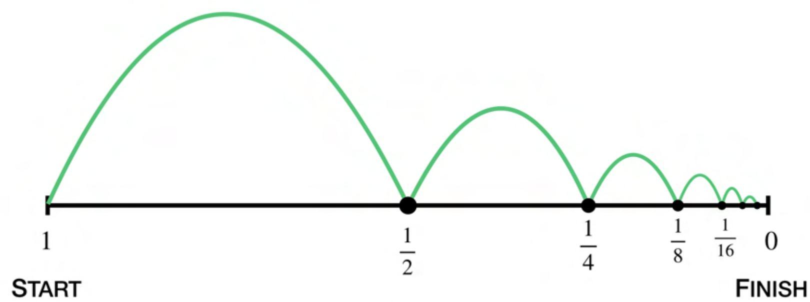


Figure 3.4: Zeno's Paradox

and the end of the path so that he is 0.25km from the end. The paradox is that Zeno must always travel half the distance from the end of the path, so he becomes *infinitely* close to the end but never reaches it.

However, we can resolve this paradox by using that his distance from the end of the path converges to zero. As Zeno walks along the path, his distance from the end of the path can be represented by the sequence $x = (1, 0.5, 0.25, 0.125, 0.0625, 0.03125\ldots)$. One way to resolve this paradox is that both the distance to the end of the path and the time needed to travel these infinitely small distances converge to zero.

Definition 4. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence and d be a metric, then x_n converges to a limit $x^\star$ if for any $\epsilon > 0$ there exists some $N > 0$ such that $d(x_n, x^\star) < \epsilon$ for all $n > N$. Denote that x_n converges to $x^\star$ by $x_n \to x^\star$ as $n \to \infty$.*

In this text, we are interested in sequences of real numbers so our metric will always be $d(x, y) = |x - y|$ for the sake of simplicity, but we could use any metric. In fact, every metric defined on the set of real numbers is equivalent in the sense for any sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \to x$ with respect to some metric d, then $x_n \to x$ with respect to any other metric defined on $\mathbb{R}$. The proof for why this fact is true is beyond the scope of this text, but would be proven in a linear algebra or functional analysis class.

Example 5. *Let $x = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ be a sequence and compute the first 5 terms of the sequence and show that $x_n \to 0$.*

Proof. To begin, the first 5 terms of this sequence are $x = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots)$. To show that this sequence converges, choose any $\epsilon > 0$ and there exists some N such that $\frac{1}{N} < \epsilon$. Then solving this inequality for N yields that when $N > \frac{1}{\epsilon}$

$$\left|x_n - 0\right| = \left|\frac{1}{n}\right| < \epsilon$$

for all $n > N$. □

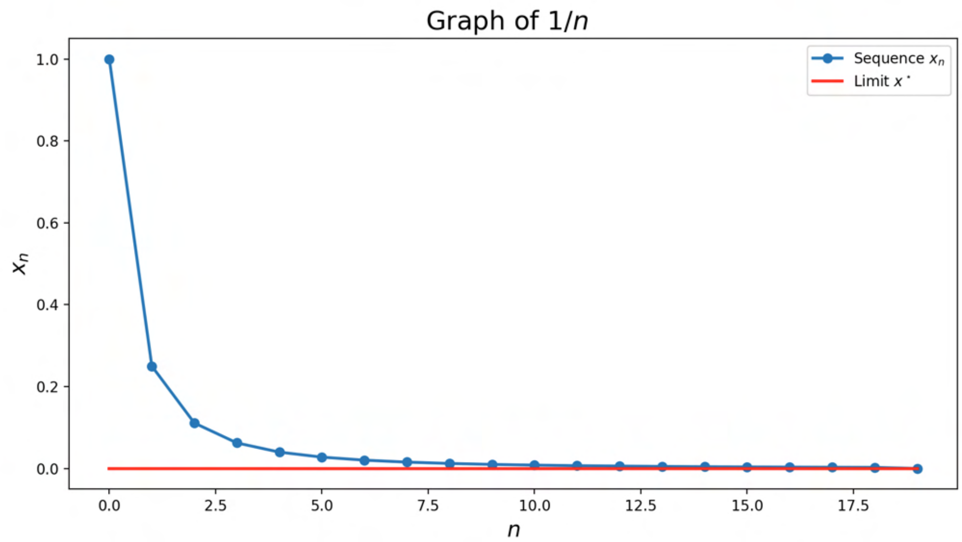


Figure 3.5: Convergent Sequence

Example 6. *Let* $\{\sin(n)\}_{n=1}^{\infty}$ *be a sequence. This sequence oscillates infinitely many times between -1 and 1 so this sequence does not converge.*

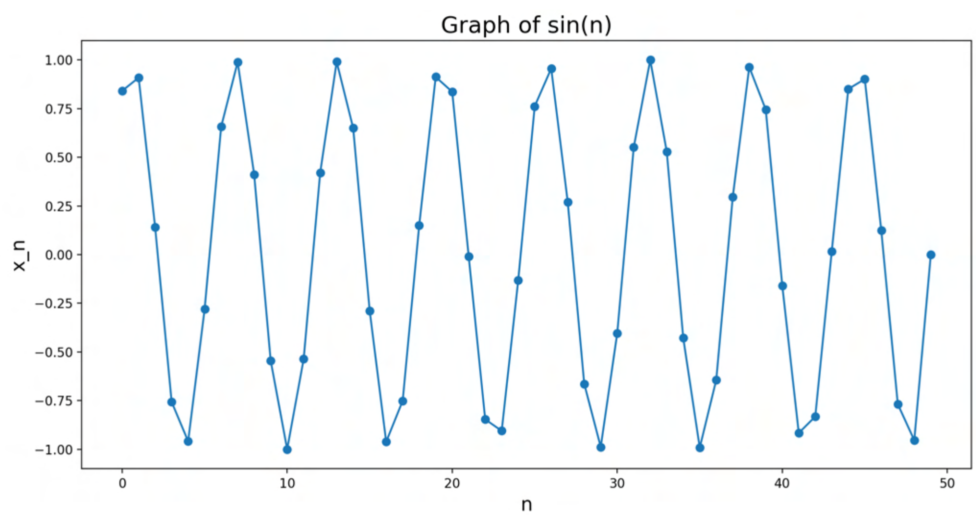


Figure 3.6: Oscillating Sequence

Example 7. *Let* $\{\frac{1}{n} \cdot \sin(n)\}_{n=1}^{\infty}$ *be a sequence. The oscillations in this sequence get smaller due to the term* $\frac{1}{n}$ *so that this sequence converges to zero.*

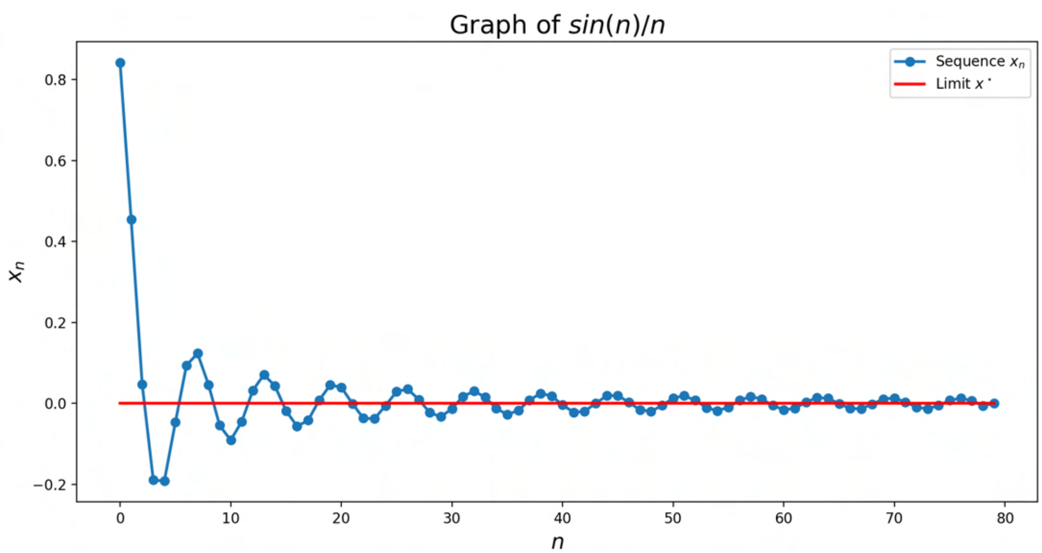


Figure 3.7: Convergent Sequence

3.3.1 Metric on the Set of Real Valued Sequences

Its natural to define a metric on spaces with an inherent geometric structure such as $\mathbb{R}$ or $\mathbb{R}^2$. But we can also define a metric on any set no matter how abstract the set may be. In this section, we define a metric on the set of all real valued sequences.

Definition 5. *Let* X *be the set of real valued sequences and define the distance function* $d : X \times X \to \mathbb{R}^+$ *to be* $d(x, y) = \sum_{n=1}^{\infty} |x_n - y_n|$ *for any* $x, y \in X$.

Proposition 1. *The distance function* d *defined in Definition 5 is a metric.*

Proof. Choose any $x, y, z \in X$, then d satisfies the following criteria of being a metric

1. It is clear that d is non-negative and when $x = y$, then $d(x, y) = 0$. If $d(x, y) = 0$, then this means that $x_n = y_n$ for all $n \in \mathbb{N}$ which implies that $x = y$.

2. Symmetry holds by

$$\begin{aligned} d(x, y) &= \sum_{n=1}^{\infty} |x_n - y_n| \\ &= \sum_{n=1}^{\infty} |y_n - x_n| \\ &= d(y, x) \end{aligned}$$

3. Lastly, the triangle inequality holds by

$$\begin{aligned}
d(x,z) &= \sum_{n=1}^{\infty} |x_n - z_n| \\
&= \sum_{n=1}^{\infty} |x_n - y_n + y_n - z_n| \\
&\leq \sum_{n=1}^{\infty} \Big(|x_n - y_n| + |y_n - z_n| \Big) \\
&=\leq \sum_{n=1}^{\infty} |x_n - y_n| + \sum_{n=1}^{\infty} |y_n - z_n| \\
&= d(x,y) + d(y,z).
\end{aligned}$$

□

Next, let's look at an example of computing the distance between two real valued sequences with respect to the metric d.

Lemma 2.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Example 8. *Let $x = \left\{\frac{1}{n}\right\}_{n=0}^{\infty}$ and $y = \left\{\frac{1+n}{n^2}\right\}_{n=0}^{\infty}$ be two sequence and compute the distance between these sequences with respect to the metric d.*

Proof.

$$\begin{aligned}
d(x,y) &= \sum_{n=1}^{\infty} \left| \frac{1+n}{n^2} - \frac{1}{n} \right| \\
&= \sum_{n=1}^{\infty} \left| \frac{1+n-n}{n^2} \right| \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= \frac{\pi^2}{6}.
\end{aligned}$$

□

Proposition 2. *Let X be the set of all real valued sequences. Define the relation $\sim$ such that $x \sim y$ for any $x, y \in X$ when $x \to y$ under the metric d, then this defines an equivalence relation on X.*

Proof.

1. Reflexivity holds because d is a metric so $d(x,x) = 0$ so $x \sim x$ for any $x \in X$.

2. Suppose $x \sim y$, then $y \sim x$ because $d(x,y) = d(y,x)$ by d being a metric.

3. Lastly, suppose that $x \sim y$ and $y \sim z$. Given that $x \to y$ and $y \to z$, then there exists some $N_x, N_y > 0$ such that $d(x, y) < \epsilon/2$ for all $n > N_x$ and $d(y, z) < \epsilon/2$ for all $n > N_y$. Then for any $n > \max(N_x, N_y)$

$$d(x, z) \leq d(x, y) + d(y, z) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds by the triangle inequality.

□

Simplified 100 Rooms Problem. *Suppose that a countably infinite number people are about to stand in a line and each is given either a red or blue hat as shown in Figure 3.8. Each person can everyone's hat who is ahead of them, but cannot see their own hat. Once everyone puts on their hat, each person must guess the color of their hat starting from the back to the front of the line. They are not allowed to communicate once they are standing in line, but they are allowed to talk beforehand and come up with a strategy. The challenge is to think of a strategy where only a finite number of people incorrectly guess the color of their hat.*

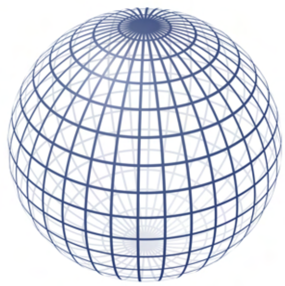
Proof. While the people are strategizing they realize that once they put on their hats, the hats form a binary sequences where blue$\to$ 0 and red$\to$ 1. They decide to define the equivalence relation that two binary sequences are equivalent if their terms are identical after some index. This equivalence relation partitions the set of binary sequences into equivalence classes. Under the assumption of the axiom of choice, they can decide on a representative for each equivalence class. Once everyone puts on their hat, each person will be able to see infinitely many terms of this sequence. This means that each person knows which equivalence class the sequence belongs to and the corresponding representative. Then the n-th person should guess the n-th term in the representative sequence. The sequence formed by the hats will eventually converge to the representative sequence, so only a finite number of people will be incorrect. □

Figure 3.8

3.4 Exercises

Metric Spaces

1. What are some examples of useful notions of distance that you could define on a sphere. You don't have to come up with a mathematical formula for the distance, just describe it in your own words.



2. Suppose we would like to define a metric on New York City, let $t : \text{NYC} \times \text{NYC} \to \mathbb{R}^+$ is a function that measures the time it takes to travel between two points in New York City. Why doesn't t satisfy the criteria of being a metric?

3. Let X be a set and show that the following function defines a metric.

$$f(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

4. Let X be the set of finite binary sequences with length n. Show that $d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$ is a metric.

5. Let $d : \mathbb{R}^2 \times \mathbb{R}^2$ be the function

$$d(x,y) = \max_{i \in \{1,2\}} |x_i - y_i|$$

with $x = (x_1, x_2)$ and $y = (y_1, y_2)$, now show that d is a metric. $\Big($Hint: you can use that $\max\limits_i |a_i + b_i| \leq \max\limits_i |a_i| + \max\limits_i |b_i|$ $\Big)$

6. (Challenge) Let X is the set of of all linear functions which means that $X = \{f(x) = ax + b : a, b, x \in \mathbb{R}\}$. Can you think of a distance function to define on this set? Can you prove that your distance function is a metric? (Hint: try drawing a picture)

Sequences

7. Let $x = \{1/2^n\}_{n=1}^{\infty}$ be a sequence and define a second sequence $s = \Big\{ \sum\limits_{k=1}^{n} x_k \Big\}_{n=1}^{\infty}$ which is referred to as the sequence of partial sums. Now compute the first 5 terms of the sequence s.

8. Determine the rule that generates each of the sequences below and provide a formula for the k-th element of the sequence.

(a) $a = (1, 9, 25, 49, \ldots)$

(b) $b = (1, 2, 2, 4, 8, 32, \ldots)$

(c) $c = (1, 1/3, 1/9, 1/27, \ldots)$

(d) $d = (1, 3, 6, 10, 15, 21, \ldots)$ $\sim$ (Challenge)

9. Suppose you are given a sequence $x = (x_1, x_2, x_3, \ldots)$, can you think of a way to partition this sequence into 10 distinct subsequences. ($\star$)[1]

10. (Challenge) Can you construct an infinite number of subsequences from a sequence?

Convergence of Sequences

11. The golden ratio can be approximated by the -sequence $r = (r_1, r_2, r_3 \ldots)$ where $r_n = x_{n+2}/x_{n+1}$ with x_n being the n-th term in the Fibonacci sequence (see Example 3). Compute the first 10 terms of the sequence r.

12. Compute the first 4 terms of the sequence of errors $e = (e_1, e_2, \ldots)$ such that $e_n = |r_n - 1.6180|$ which is the difference between each term in the sequence and the golden ratio. Lastly, plot the sequence of errors where the x-axis is the index and y-axis is the error.

13. Let x be a sequence such that its terms are determined by the formula $x_n = 2 - \frac{1}{n}$.

    (a) Determine the limit of this sequence, it might be helpful to compute the first few terms of this sequence.

    (b) Suppose that you are given $\epsilon_1 = 1, \epsilon_2 = 1/10$, and $\epsilon_3 = 1/1000$, now determine for what value of N guarantees that $|x_n - x^\star| < \epsilon_i$ for all $n > N$ for each value of epsilon.

    (c) Now suppose that you are given any $\epsilon > 0$, then what value of N guarantees that $|x_n - x^\star| < \epsilon_i$ for all $n > N$.

14. Let $x_n = 1 + \frac{2}{n}$ be a sequence, then compute the first 5 terms and show that $x_n \to 1$.

15. (Challenge) Let $x_n = \frac{1}{n}\sin(n)$ be a sequence, then compute the first 5 terms and show that $x_n \to 0$.

16. Let $x_n = 2 + (-1)^n$ be a sequence and compute the first 5 terms of this sequence. Can you find two convergent subsequences?

17. Let X be the set of all real valued sequences. Show that $\sim$ is an equivalence relation such that $x \sim y$ for $x, y \in X$ if there exists an $N > 0$ such that $x_n = y_n$ for all $n > N$. ($\star$)

Logic Puzzle

18. Suppose that an infinite number of people are about to go into a room and each puts on a different hat. Each hat has a real number on it and everyone can see the real numbers on all of the hats except their own. After all the hats are placed on, everyone has to simultaneously

[1] Result used to solve 100 rooms problem

shout out what real number they think is on their hat. They are not allowed to communicate once they enter the room, but beforehand they are allowed to talk and come up with a strategy. The challenge is to think of a strategy where only a finite number of people incorrectly guess the number on their hat. $(\star)$

Chapter 4

100 Rooms Problem

Suppose there are 100 rooms and each room has the exact same infinite sequence of boxes with each box containing a real valued number. There are 100 people and each person is assigned to one of the 100 rooms. After everyone has entered their room, each person must guess what number is in at least one of the boxes. Each person may open as many boxes as they want, even infinitely many but must leave at least one box unopened. Before everyone enters their room, the group of people may decide on a strategy. The challenge is to find a strategy such that 99 people correctly guess what number is in at least one box.

Let's do a concrete example with only two rooms and illustrate the solution in three steps before we go through the solution of the 100 Rooms Problem.

4.1 Simplified Problem

Before entering the rooms, you realize that the infinite sequence of boxes in each room can be thought of as a sequence because each box contains a real valued number. Now define the equivalence relation on the set of real valued sequences such that $x \sim y$ if there exists some index k such that $x_k = y_k$ for all $k > K$. (Note: we proved this in Exercise 17 in Section 3.4).

Step 1

Suppose that you enter Room 1 and so you immediately rearrange all of the boxes into the two subsequences as illustrated in Figure 4.1. Then let b_1 and b_2 denote the resulting subsequences.

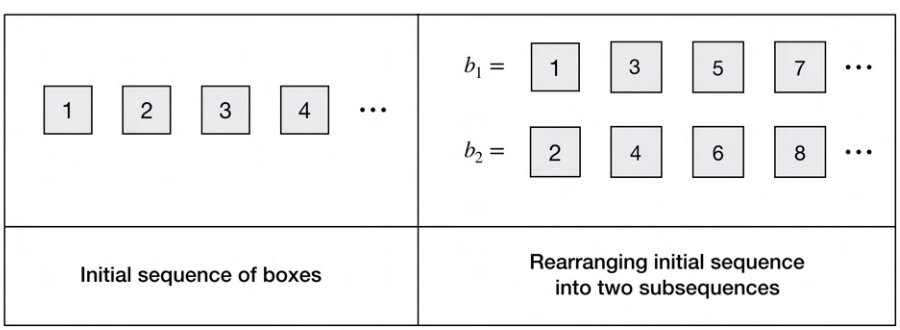


Figure 4.1: Step 1

Step 2

Now open all of the boxes in the subsequence b_2 so that you can determine which equivalence class the sequence belongs to and recall the representative of this equivalence class. In the example in Figure 4.2, the subsequence b_2 converged to the representative sequence by the 4th box.

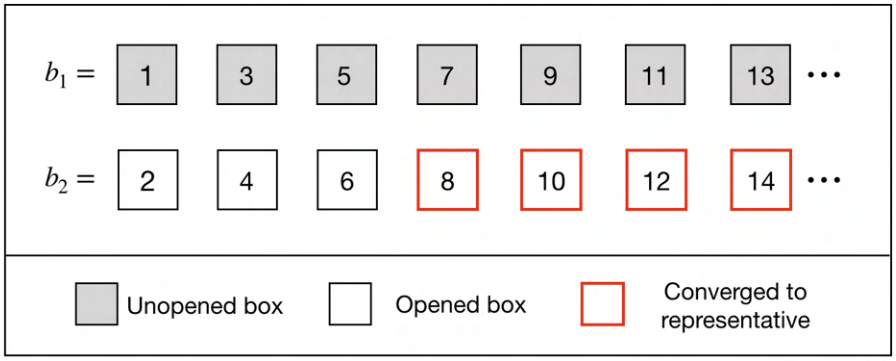


Figure 4.2: Step 2

Step 3

Now open all of the boxes in the subsequence b_1 except for the first 5 boxes. Then you can determine which equivalence class b_1 belongs to and recall the representative of this equivalence class, which let's denote as $b_1^\star$. Lastly, you should guess that the number in the 5th box in the subsequence b_1 is the 5th term in the representative sequence $b_1^\star$. If b_1 converges to the representative before b_2 converges to its representative, then your guess will be correct. Otherwise, if the person in Room 2 follows this procedure, their guess will be correct.

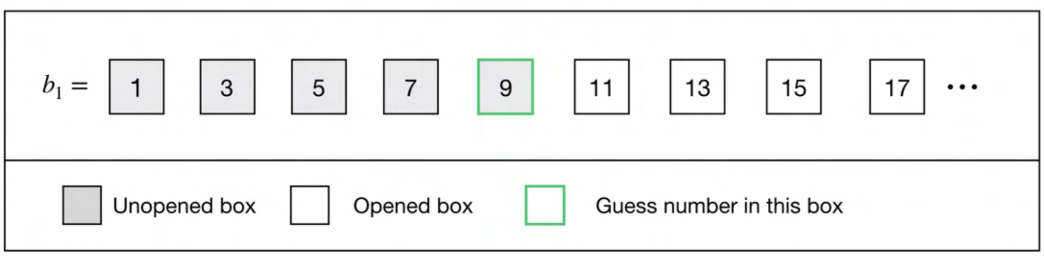


Figure 4.3: Step 3

4.2 General Problem

Now let's adapt our solution from the previous subsection to the general case of 100 rooms.

1. **Define an Equivalence Relation**

   Before entering the rooms, you realize that the infinite sequence of boxes in each room can be thought of as a sequence because each box contains a real valued number. Now define the equivalence relation on the set of real valued sequences such that $x \sim y$ if there exists some index k such that $x_k = y_k$ for all $k > K$. (Note: we proved this in Exercise 17 in Section 3.4).

2. **Equivalence Classes and the Axiom of Choice**

   This equivalence relation partitions the set of real valued sequences into equivalence classes. Using the Axiom of Choice, the group chooses a representative from each equivalence class.

3. **Partition Sequence into 100 Subsequences**

   Once each person enters their room, they rearrange the sequence of boxes into 100 subsequences. The first subsequence is $b_1 = \{$ box 1, box 101, box 201,...$\}$ and the n-th subsequence of boxes is $b_n = \{$ box n, box 100+n, box 200+n,...$\}$. We we provide a visualization of the 100 subsequences in Figure 4.1, where each square represents a box and the number is the index.

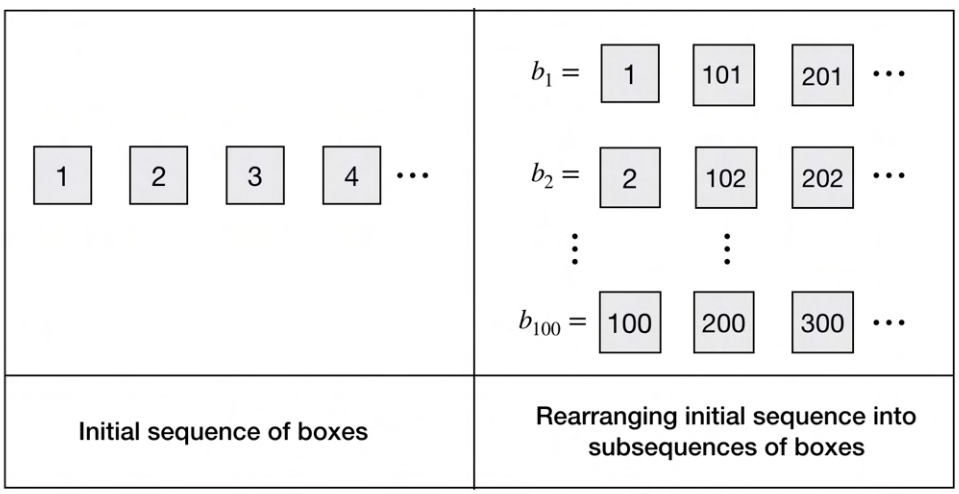


Figure 4.4

4. **Open Boxes**

   Now the person in room n opens every box in every subsequence except for the subsequence b_n. For example, the person who enters room 5 will open every single box except for the boxes in the subsequence $b_5 = \{5, 105, 205, 305, \ldots\}$.

5. **Determine the Equivalence Class of Each Subsequence**

   When the person in room n opens all of the boxes in a given subsequence b_k such that $k \neq n$, they uncover a real valued sequence. This person sees infinitely many terms in the subsequence so they can also determine which equivalence class the sequence belongs to. Now this person needs to recall the representative of this equivalence class and denote this representative by $b_k^\star$.

6. **Record Index of Convergence**

   Let $b_k(m)$ denote the m-th term in the sequence b_k. Then given that b_k and $b_k^\star$ belong to the same equivalence class, there exists some index M_k such that $b_k(m) = b_k^\star(m)$ for all $m > M_k$. The value of M_k is the index where b_k converges to the representative of its corresponding equivalence class. The person in room n needs to find this index for every subsequence except b_n and record the maximum index, namely $M^\star = \max\big\{M_k : k \in \{1, \ldots, 100\} \backslash n\big\}$

7. **Guess Real Number**

   Now this person opens all of the boxes in the subsequence b_n except for the first $M^\star$ boxes. Once all the boxes have been opened, the person can determine equivalence class the sequence b_n belongs to and recall the representative $b_n^\star$. Now this person should guess that the number in the $(M^\star{+}1)$-th box is the number $b_n^\star(M^\star{+}1)$. As long as $M_n^\star < \max\{M_k : k \in \{1, \ldots, 100\}$, then this guess will be correct because the sequence b_n would have already converged to the representative.

Chapter 5

Logic Problems

5.1 Hilbert Hotel

Problem 1: Finite Number of Guests Arrive

Suppose that n new guests arrive at the Hilbert hotel. Design a scheme to accommodate the newly arrived guests.

Problem 2: Infinite Number of Guests Arrive

Suppose that an infinite number of guests arrive at the Hilbert Hotel. Design a scheme to accommodate the newly arrived guests

Problem 3: Finite Number of Carriages each with Infinitely Many Guests Arrive

Suppose a finite number of carriages such that each contains an infinite number of guests arrive at the Hilbert Hotel. Design a scheme to accommodate the newly arrived guests

5.2 Blue and Red Hats

Problem 4: 100 People Standing in a Line

Suppose that 100 people are about to stand in a line and each puts on either a red or blue hat. Everyone is facing the same direction as shown in Figure 3.8, so each person can only see the color of the hats of the people standing ahead of them. Once everyone puts on their hat, each person has to guess the color of their hat starting from the back to the front of the line. The players win if only 99 people can correctly guess the color of their hat. They are not allowed to communicate once they are standing in line, but beforehand they are allowed to talk and come up with a strategy.

Problem 5. Two Hat Problem

Suppose that you are in a room with one other person and either a white or black hat will be placed on your head in a few moments. This means that each player can see the other players hat, but cannot see their own hat. Once the hats are placed on each person's head, then both players have to simultaneously guess the color of their hat. If at least one player correctly guesses the color, then both players win and otherwise both players lose. The players may decide on a strategy before entering the room, but there is no communication once they've entered the room.

Problem 6. Three Hats Around a Table

Suppose that you are sitting around a table with two other people. You are told that there are three red and two blue hats and each person will be given a hat. You won't see the left over hats, but you do see that the two other people both have red hats. The winner is the person who correctly guesses the color of their hat. Once the game begins, the two other people are unwilling to guess the color of their hat. What color is your hat?

Problem 7: Four Hats and a Wall

Suppose that four people are standing in a line and all facing the same direction, but the fourth person is separated from the other three by a wall. The first is wearing a red hat, the second a blue hat, the third a red hat, and the fourth a blue hat. They are told that there are two red hats and two blue hats, but they can only see the hats on people standing in front of them. The person who shouts out the color of their hat wins, so who will win?

Problem 8: Infinitely Many People Standing in a Line

Assume the same set up as the previous problem, except that now there are an infinite number of people standing in line. The players win if only a finite number of people incorrectly guess the color of their hat.

5.3 Jacks, Jokers, and Spies

Problem 9: Approached by Two People

Suppose that you are on an Island, where Jacks always tell the truth and Jokers always lie. You are approached by two people and you know that one of them must be a Jack and the other a Joker. The first says that they are both Jokers, but are they actually? [1]

Problem 10: Approached by Two People, Again

Now suppose that you are approached by two people such that one is a Jack and the other is a Joker. The first points to the second and says that he is a Joker. The second one says neither of us are Jokers. What are they actually? [2]

Problem 11: Fork in the Road

Suppose that you come to a fork in the road with one man standing before each path. One path leads to freedom and the other to death. You know that one is a Joker and the other is a Jack. You can ask one of the men one question to determine which path leads to freedom. [3]

Problem 12: Approached by Three People

Suppose that you approached by three people such that there is one Jack, Joker, and spy who sometimes tells the truth. The man in blue says, "I am a Jack" The man in red says, "He speaks the truth" The man in green says, "I am a spy". Who is the Jack, who is the Joker, and who is the spy? [4]

Problem 13: Approached by Three People, Again

Suppose that you approached by three people such that there is one Jack, Joker, and spy who sometimes tells the truth. You can ask these them two yes-or-no questions. They will all answer *you, one at a time, with either "yes" or "no". Who is the Jack, who is the Joker, and who is the spy?* [5]

5.4 Additional Problems

Problem 14: Monty Hall Problem

Suppose that you are playing a game where there are three doors such that a goat is behind two of the doors and a brand new car is behind one of the doors. At the start of the game, you must pick one of the doors and soon after the host opens one of the two doors that you did not pick to reveal that there is a goat behind this door. Now there are two doors that are unopened and the host gives you the option of either keeping your original door or switching your guess to the other unopened door. What is the best strategy and why?

Problem 15: Odd One Out

Suppose there are 12 people and 11 of the people have the exact same weight, but one person weighs a little be more and the goal is to guess this person. You have a balance and can put any number of peoples on either end of the balance, but you can only use the balance three times. It is possible to come up with a solution where you can correctly guess the person.

Problem 16: The 100 Coins

There are 10 sets of 10 coins and you know that all of the coins in nine of the sets weigh an ounce. Every coin in one of the sets is off by exactly a hundredth of an ounce, which means that one of the sets is off by a tenth of an ounce. You are allowed to use a digital weighing machine only once, so how can you determine which set of coins is faulty.

Problem 17: Egg Drop

Suppose that you have two eggs and you are on a 100 story building. The challenge is to determine at exactly what floor the egg will break. You are allowed 20 trials and after that you will be given two eggs to prove that you have determined the correct floor.

Problem 18: Hourglass

Suppose that you have an 11-minute hourglass and a 7-minute hourglass. How can you measure exactly 15 minutes?

Problem 19: Burning Rope

Suppose that you are given two ropes and a lighter such that if you light one end of the rope, it takes exactly one hour to burn to the other end. The rope doesn?t necessarily burn at a uniform rate. How can you measure a period of 45 minutes?

Problem 20: Landscaping

Suppose that you are a landscaper and have been asked to design a garden with four trees that are all equidistant from each other. How do you place the trees?

Problem 21: Crossing a Rickety Bridge

Four people need to cross a rickety bridge at night. Unfortunately, they have only one torch and the bridge is too dangerous to cross without one. The bridge is only strong enough to support two *people at a time. Not all people take the same time to cross the bridge. Times for each person: 1 min, 2 mins, 7 mins and 10 mins. What is the shortest time needed for all four of them to cross the bridge?*

Problem 22: Light Switches

Suppose that there is a light bulb in a closet with the door closed and the light turned off. There are three light switches outside the door and only one of them controls the light in the closet. You may flip the switches as many times as you want, but you cannot touch any of the switches once you open the door. How can you figure out which light switch controls the light?

Problem 23: Three Light Switches and Three Light Bulbs

You are in a room with 3 switches which correspond to 3 bulbs in another room and you don?t know which switch corresponds to which bulb. You can only enter to the room with the bulbs and back once. How do you find out which bulb corresponds to which switch?

Problem 24: River Crossing

Suppose there are 3 foxes and 3 bunnies on one side of a river. The river is infinitely long and there is no bridge, so the only way to cross the river is by boat. There is a small boat that can only hold three animals. To keep the bunnies safe, there can never be more fox than bunnies. Can you come up with a strategy so that all the animals can cross the river.

Problem 25: Ramsey Number

Suppose that you draw 6 points and connect every single point with either a blue or red line. Then no matter how you color the lines, you can always find three points along with their corresponding edges that forms either a blue or red triangle.

Problem 26: Poisoned Bottles of Wine $\star$

Suppose that a King throws a party and invites 1000 guests who each bring the King a bottle of wine. Assume that the Queen discovers that one of the guests is trying to assassinate the King by giving him a poisoned bottle of wine. However, they do know which guest or which bottle of wine is poisoned. Now the King also has 10 prisoners that he plans to execute and he decides to use them as the taste tester. The poison when taken has no effect on the prisoner until exactly 24 hours later when the infected prisoner suddenly dies. The King needs to determine which bottle of wine is poisoned by tomorrow so that the party can continue as planned. How can the King administer the wine to the prisoners to ensure that 24 hours from now he is guaranteed to have found the poisoned wine bottle?

Problem 27: Equilateral Triangle in the Plane after Random Coloring $\star$

Suppose that every point $(x, y) \in \mathbb{Z}^2$ *is colored either red or blue and that the probability of a point being either red or blue is exactly 0.5. Let* $\mathcal{C} : \mathbb{Z}^2 \to \{\textit{red, blue}\}$ *be a function that returns the color of any point in the plane. For example, suppose that the point* (x, y) *is red, then* $\mathcal{C}(x, y) = \textit{red}$*. Prove that there exists at least three point in the plane that have the same color and form the vertices of an equilateral triangle.*

Chapter 6

Solutions

6.1 Hilbert Hotel

Problem 1: Finite Number of Guests Arrive

Proof. To accommodate the new guests, move the guest in room 1 to room $n+1$, the guest in room 2 to $n+2$,... and etc. Then the newly arrived guests can be assigned to the first n rooms. $\square$

Problem 2: Infinite Number of Guests Arrive

Proof. To accommodate the new guests, move the guest in room 1 to room 2, the guest in 2 to room 4, the guest in room 3 to room 6 so the rule for moving the guests is that the guest in room n moves to room $2n$. This means that the current guests will be moved to the even numbered rooms, which implies that all of the odd numbered rooms rooms will be unoccupied. This means that we can move the n-th newly arrived guest to the $2n-1$-th room. $\square$

Problem 3: Finite Number of Carriages each with Infinitely Many Guests Arrive

Proof. Suppose that n carriages arrive and denote the k-th person in the j-th carriage by the pair (j,k). Now let's use the convention that the 0-th carriage corresponds to the current guests at the Hilbert Hotel. Now let's use the rule that $(j,k) \mapsto (k \cdot n + j)$-th room for all $j,k,n \in \mathbb{N}$. $\square$

6.2 Blue and Red Hats

Problem 4: 100 People Standing in a Line

Proof. The key to this problem is that the first person in the line can see everyone's hat except their own. Given that the first person can see 99 hats, the strategy is to count the number of red hats. Before the players stand in line, the group decides that if the first person says *red*, then this means that the first person counted an *even* number of red hats. Without loss, suppose that the first person says that their hat is red, then the second person in line knows that the first person saw an even number of red hats. Now the second person counts the number of red hats that they can see. If this number is still even, then their hat must be red and otherwise their hat must be blue. Each person proceeds in this manner which means that the last 99 people will correctly guess their hat. ☐

Problem 5: Two Hat Problem

Proof. Given that the hats can only be one of two different colors, then this means that players will either have the same or different colored hats. The strategy is that player one guesses that their hat is the color of the second person's hat and player two guesses the opposite of the first players hat. If the players have the same colored hat, then player one will be correct. Otherwise, if the players' hats are different colors, then player two will be correct. ☐

Problem 6. Three Hats Around a Table

Hint: Try looking at the problem from the other people's perspective.

Problem 7: Four Hats and a Wall

Hint: Solution is similar to previous problem.

Problem 8: Infinitely Many People Standing in a Line

Proof. While the players are strategizing they realize that once they put on their hats, then their hats will form a binary sequences where blue$\rightarrow$ 0 and red$\rightarrow$ 1. They decide to define the equivalence relation that two binary sequences are equivalent if their terms are identical after some index. This equivalence relation partitions the set of binary sequences into equivalence classes. Under the assumption of the axiom of choice, they can decide on a representative for each equivalence class. Once everyone puts on their hat, each person will be able to see the tail of the sequence and determine which equivalence class the sequence belongs to and determine the representative. This means that the n-th person should guess the n-th term in the representative sequence. The sequence formed by the hats will eventually converge to the representative sequence, which means that only a finite number of people will be incorrect. ☐

6.3 Jacks and Jokers

Problem 9: Approached by Two People

Proof. Since you know that one person must be a Jack and the other a Joker, then this person must be lying since there can't be two Jokers. So the person who spoke must be a Joker and the other is the Jack. ☐

Problem 10: Approached by Two People, Again

Hint: Solution is similar to previous problem.

Problem 11: Fork in the Road

Hint: Ask if the other person would tell you to go down the other person's path.

Problem 12: Approached by Three People

Problem 13: Approached by Three People, Again

6.4 Additional Problems

Problem 14: Monty Hall Problem

Proof. When you initially choose a door, the probability that you chose the door with the car is $0.\overline{3}$. This implies that the probability that the car is behind one of the two remaining doors is $0.\overline{6}$. The host knows which door the car is behind, so he uses his knowledge to open one of the unchosen doors to reveal a goat. Now the probability that the car is behind you door is still $0.\overline{3}$, but the probability that the car is behind the other door is $0.\overline{6}$ so you should choose this door. $\square$

Problem 15: Odd One Out

Proof. First, split the group into two groups of six and place each group at one end of the balance. Since one person weighs a little more than the others, then one end of the balance will tip downwards and keep the people on this end. Now there are six people, so place three on one end and three on the other end. Again, one end will tip downward and keep those three people. Lastly, choose any of the two remaining people and place each at one end of the scale. If the balance tips in one direction, then we immediately know that this person weighs a little more than the rest. Otherwise, if they are the same weight, then the balance will not tip towards either. This means the one person who is not on the balance weighs a little more than the rest. $\square$

Problem 16: The 100 Coins

Proof. The strategy is to make a set of coins which contains one coin from the first set, two from the second set, three from the third set, and etc. If all of the coins had the exact same weight, then this new set we constructed would weigh 55 ounces. Suppose that set n is the one that is off by a tenth of an ounce, then our new set will be off by $n/100$-ths of an ounce. $\square$

Problem 17: Egg Drop

Hint: Use binary search

Problem 18: Hourglass

Proof. To begin, flip over both hour glasses and let both run until the seven minute hourglass runs out so that the 11 minute one has 4 minutes until its empty. Now you can measure 15 minutes by using the four minutes remaining in the 11 minute hour glass. Once this time has run out, the 11 minute hour glass will be empty so flip it and let it run for 11 minutes. $\square$

Problem 19: Burning Rope

Solution by Tim Gore, Rodrigo Guerreiro, and Jiahe Ling

Proof. Although the rope may not burn at a uniform rate, the entire rope burns in 60 minutes. If we light both ends of the rope at the same time, the entire rope will burn in 30 minutes and we can use this technique to measure 30 minutes. Next, we can we the second rope to measure 15 minutes. If we light one end of the second rope at the same instant that the first is lit, the second rope will have burned for 30 minutes after the first rope burns up. Now whatever is left of the second rope will in 30 minutes, so light both ends so that the remaining rope burns in 15 minutes. Thus, when the second rope burns up, 45 minutes will have passed. □

Problem 20: Landscaping

Hint: Plant the trees on a hill.

Problem 21: Crossing a Rickety Bridge

Solution by Wilton Bompey, Kolia Krajewski, and Cedric Sirianni

Proof.

Trip 1: 1 and 2 minute people cross the bridge together (2 mins, 2 mins total)
Trip 2: 1 minute person goes back (1 min, 3 mins total)
Trip 3: 10 and 7 minute person cross the bridge (10 mins, 13 mins total)
Trip 4: 2 minute person goes back (2 mins, 15 mins total)
Trip 5: 2 and 1 minute person cross the bridge for the final time (2 mins, 17 mins total)

□

Problem 22: Light Switches

Proof. First, flip on the first light switch and leave it on for at least a few minutes, then flip off the first switch and the second on. Now enter the room and if the light is turned on, then the second switch must control this light. Otherwise if the room is dark, then the first switch controls the light bulb if it feels warm and the second switch if its cold. □

Problem 23: Three Light Switches and Three Light Bulbs

Hint: Similar to previous problem

Problem 24: River Crossing

Hint: Solution is similar to the Crossing a Rickety Bridge Problem

Problem 25: Ramsey Number

Hint: Try to find a configuration that does not contain a monotone triangle.

Problem 26: Poisoned Bottles of Wine $\star$

Problem 27: Equilateral Triangle in the Plane after Random Coloring $\star$

Proof. Suppose we consider all of the points on the x-axis, then there must be either an infinite number of points that are red or an infinite number of points that are blue by the pigeon hole principle. Without loss, assume that there are an infinite number of points that are red and define this set to be $R = \{(x, 0) \in \mathbb{Z}^2 : \mathcal{C}(x, 0) = \text{red}\}$. The set R is countable because $R \subset \mathbb{Z}^2$, then this set can be enumerated as $R = \cup_{i \in \mathbb{N}} x_i$. For any adjacent points say x_n and x_{n+1}, there is a third point y_n such that these three points are equidistant. This means that if y_n is red, then we have found an equilateral triangle.

Let $\mathbb{P}(\Delta_{1:n})$ be the probability of the points x_i, x_{i+1}, and y_i forming a red triangle for any $1 \leq i \leq n$. It is clear that $\mathbb{P}(\Delta_i) = 0.5$ for any i, the probability that one of y_1 and y_2 being red is

$$\begin{aligned}\mathbb{P}(\Delta_{1:2}) &= 1 - \mathbb{P}(\Delta_1) \cdot \mathbb{P}(\Delta_2) \\ &= 1 - \frac{1}{2^2} \\ &= \frac{3}{4}.\end{aligned}$$

Given that we have the sequence $\{y_n\}_{n=1}^{\infty}$, then the probability of at least one of the first k terms being red is

$$\begin{aligned}\mathbb{P}(\Delta_{1:k}) &= 1 - \prod_{n=1}^{k} \mathbb{P}(\Delta_n) \\ &= 1 - 0.5^k.\end{aligned}$$

But then as $k \to \infty$ then $\mathbb{P}(\Delta_{1:k}) \to 1$.

$\square$

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